

STOCHASTIC EXTENDED PATH APPROACH

STÉPHANE ADJEMIAN AND MICHEL JUILLARD

ABSTRACT. The Extended Path (EP) approach is known to provide a simple and fairly accurate solution to large scaled nonlinear models. The main drawback of the EP approach is that the Jensen inequality is neglected, because future shocks are (deterministically) set to their expected value of zero. Previous contributions shown that the cost of this approximation is small compared to the cost of neglecting the deterministic nonlinearities. But the accuracy errors are significantly increased in the presence of binding constraints (such as a Zero Lower Bound on nominal interest rates). In this paper we propose a simple extension to the EP approach by considering that the structural innovations in $t+1$ are non zero and keeping the innovations in $t+s$ ($s>1$) equal to their expected value of zero. We use a quadrature approach to compute the expectations under this assumption. We evaluate the accuracy of the Stochastic Extended Path approach on a Real Business Cycle model. The computing time of this approach is polynomial in the number of endogenous variables but exponential with respect to the number of structural innovations.

1. INTRODUCTION

The aim of this paper is to improve the Extended Path approach when solving a non linear model with occasionally binding constraints.

The extended path approach relies on a perfect foresight solver to take full account of the non-linearities introduced by the occasionally binding constraints. For each period of the sample, contemporaneous exogenous innovations are treated as surprise shocks in a deterministic simulation where shocks are set to their expected value of zero in all future periods. This approach neglects the Jensen inequality, but [Adjemian and Juillard \(2011\)](#) considered that it was a minor drawback in comparison with the correct treatment of the non-linearities induced by the zero lower bound.

Few studies, *e.g.* [Gagnon \(1990\)](#) and [Love \(2009\)](#), evaluate the accuracy of this simulation method. These authors, considering a stochastic growth model, show that the approximation errors are reasonable and that the extended path approach performs as well (or even better) as a global approximation approach. However the degree of non linearity of the stochastic growth model with a Cobb-Douglas technology is relatively weak and they do not consider models with occasionally binding constraints.

In a previous contribution, *e.g.* [Adjemian and Juillard \(2011\)](#), we evaluated the accuracy of the EP approach when simulating a New-Keynesian model with [Calvo \(1983\)](#) nominal rigidity on

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prices, [Kimball \(1996\)](#) aggregation function of intermediate goods *and* a Zero Lower Bound on nominal interest rates. We found accuracy errors as small as in [Gagnon \(1990\)](#) and [Love \(2009\)](#) when the ZLB is not binding, but our main result is that the accuracy is sensibly degraded when the nominal interest rate hits the zero lower bound.

We propose a simple extension to the EP approach by considering that the structural innovations in $t + s$, $s = 1, \dots, S$ are non zero and keeping the innovations in $t + s$ ($s > S$) equal to their expected value of zero. We use a quadrature approach to compute the expectations under this assumption. We evaluate the accuracy of the Stochastic Extended Path approach on two models: a standard Real Business Cycle model and an RBC model with irreversible investment. We show that the accuracy errors are significantly reduced when $S \geq 1$. The computing time of this approach is polynomial in the number of endogenous variables but exponential with respect to the number of structural innovations.

In section 4 we present the models considered to evaluate the accuracy of the extended path approach. The simulation method is presented in section 2 and the accuracy checks are introduced in section 5.

2. STOCHASTIC EXTENDED PATH APPROACH

Nonlinear stochastic equilibrium model may be represented generally as follows:

$$(1a) \quad s_t = Q(s_{t-1}, u_t)$$

$$(1b) \quad F(y_t, x_t, s_t, \mathbb{E}_t[\mathcal{E}_{t+1}]) = 0$$

$$(1c) \quad G(y_t, x_{t+1}, x_t, s_t) = 0$$

$$(1d) \quad \mathcal{E}_t = H(y_t, x_t, s_t)$$

where s_t is a $n_s \times 1$ vector of exogenous variables, the innovation u_t is a multivariate random variable in \mathbb{R}^{n_u} with expectation 0 and variance Σ_u (the cumulative distribution function of u is denoted $P(u)$), x_t is a $n_x \times 1$ vector of endogenous state variables, y_t is a $n_y \times 1$ vector of non predetermined variables and \mathcal{E}_t is a $n_e \times 1$ vector of auxiliary variables. Q , F , G and H are non linear continuous functions (not necessarily differentiable everywhere).

2.1. EP algorithm. The extended path algorithm (EP hereafter) is a simulation approach for generating time series for the endogenous variables $\{y_t, x_{t+1}\}_{t=1}^T$ given an initial condition for the state variables, (s_0, x_1) , and a sequence of innovations $\{u_t\}_{t=1}^T$. The extended path approach indirectly characterizes the decision rules (*ie* the functions relating the non predetermined variables, y_t , with the state variables, x_t and s_t) by generating time-series for the endogenous

variables. Basically, the trick is to assume that all the agents believe that the innovations will be always zero in the future. Given the state of the economy at date t , (s_t, x_t) , we can then solve (1). Here is a sketch of the algorithm:

Algorithm 1 Extended path algorithm

1. $H \leftarrow$ Set the horizon of the perfect foresight models.
 2. $(s_1, x_1) \leftarrow$ Choose an initial condition for the state variables.
 3. **for** $t = 1$ to T **do**
 4. $(y_t, x_{t+1}) \leftarrow$ Solve a perfect foresight model with terminal condition $y_{t+H} = y^*$.
 5. $v \leftarrow$ Draw independent uniform variates $(n_s \times 1)$.
 6. $u \leftarrow P^{-1}(v)$
 7. $s_{t+1} \leftarrow Q(s_t, u)$
 8. **end for**
-

Note that, to be consistent with our notations, the initial condition should be (s_0, x_1) , whereas in step 2 of algorithm 1 we set (s_1, x_1) as the initial condition. This point incorporates the effect of the contemporaneous innovations (u_t) .

The main advantage of this approach is that we can simulate large models *with an arbitrary precision*, because the number of needed operations increases polynomially with the number of endogenous variables (the main task when solving the perfect foresight model consist in inverting a sparse matrix) and not exponentially (as it would with a global approximation of the policy rules). The extended path approach does not suffer from the so called curse of dimensionality. A second advantage is that the EP approach does not require any special treatment when the model admit occasionally binding constraints, because it does not impose the differentiability of F or G .

Obviously these advantages come at a cost: with the EP approach we abstract from the effects of uncertainty on the behavior of the agents (by assuming that the agents believe that the innovations of the exogenous states will be zero in the future we abstract from the effects of uncertainty about the future). However, two points are worth noting. First, large scaled models are usually solved considering a first order Taylor approximation of Q , F , G and H in (1). If we linearize the model, we will also neglect the effects of uncertainty about the future *and* we will not be able to treat the occasionally binding constraints. In this respect the EP approach dominates the first order perturbation approach. We could instead consider a k -order perturbation approach. If k is greater than one, the certainty equivalence property is not satisfied and uncertainty about the future has an impact on agents decisions. Nevertheless, because this approach requires the differentiability everywhere of the model, we would not be able to treat correctly occasionally binding constraints. Second, if we agree with Lucas (1987, 2003) that the cost of fluctuations is very small, it is most likely that the error of approximation induced by our approximation about future innovations is small. Indeed, Gagnon (1990),

Love (2009) and Adjemian and Juillard (2011) show that the accuracy errors are fairly small, as long as the economy does not hit an occasionally binding constraint. When the economy hits an occasionally binding constraint the household do not have the possibility to smooth their consumption levels, so that fluctuations have a bigger impact on welfare. In this situation the assumption underlying the EP approach is more problematic. In the next subsection we show how to (partially) relax this assumption about the future innovations.

2.2. Stochastic EP algorithm. To allow for non zero shocks in periods $t + 1, t + 2, \dots, t + S$ ($S \geq 1$), we need to explicitly evaluate the (conditional) expectations appearing in (1b). We use Gaussian quadrature. Suppose that X is a Gaussian random variable with mean zero and variance $\sigma_x^2 > 0$, and that we need to evaluate $\mathbb{E}[\varphi(X)]$, where φ is a continuous function. By definitions of the expectation and the Gaussian probability density function, we have $\mathbb{E}[\varphi(X)] = \frac{1}{\sigma_x \sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-\frac{x^2}{2\sigma_x^2}} dx$. This integral can be approximated using the following result (see Judd (1998)):

$$\int_{-\infty}^{\infty} \varphi(x) e^{-x^2} dx = \sum_{i=1}^n \omega_i \varphi(x_i) + \frac{n! \sqrt{n} \phi^{(2n)}(\tilde{\zeta})}{2^n (2n)!}$$

for any $\tilde{\zeta} \in \mathbb{R}$, where the last term on the right hand side is the approximation error, x_i ($i = 1, \dots, n$) are the roots of an order n Hermite polynomial, and the weights ω_i are positive. For a given order of approximation n , the approximation error is proportional to the order $2n$ derivate of the function to be integrated. This results tells us that is possible to find out a sequence of weights ω_i such that the evaluation of the integral with the sum of the right hand side is exact for any order $2n - 1$ polynomial. Golub and Welsch (1969) describe how to calculate the quadrature weights and nodes (ω_i, x_i) by computing the eigenvalues and eigenvectors of a symmetric tridiagonal matrix. Obviously a change of variable is needed to evaluate $\mathbb{E}[\varphi(X)]$. We define $z = x/\sigma_x \sqrt{2}$, and consider the following approximation for the expectation:

$$\mathbb{E}[\varphi(X)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^n \omega_i \varphi(z_i)$$

If X is a multivariate Gaussian random variable we use a Tensor product approach. For instance if X is defined in \mathbb{R}^m , $\mathbb{E}[X] = 0$, $\mathbb{V}[X] = \Sigma$, and $\psi(\mathbf{x})$ is a function from \mathbb{R}^m to \mathbb{R}^q , we use the following approximation:

$$\begin{aligned} \mathbb{E}[\psi(X)] &= (2\pi)^{-\frac{m}{2}} \Sigma^{-\frac{1}{2}} \int_{\mathbb{R}^m} \psi(\mathbf{x}) e^{-\frac{1}{2} \mathbf{x}' \Sigma^{-1} \mathbf{x}} d\mathbf{x} \\ &\approx \pi^{-\frac{m}{2}} \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_m=1}^n \omega_{i_1} \omega_{i_2} \dots \omega_{i_m} \psi(z_{i_1}^1, z_{i_2}^2, \dots, z_{i_m}^m) \end{aligned}$$

with the change of variables $\mathbf{z} \equiv (z^1, z^2, \dots, z^q)' = \Sigma^{-\frac{1}{2}} \mathbf{x} / \sqrt{2}$. The drawback of this tensor product rule is that the number of function ψ evaluations is exponential with respect to the dimension of X . As a less expensive alternative, we could consider monomial rules or the use

of sparse grids.

We use this Gaussian quadrature to approximate the expectation in the Euler equations (1b). Suppose that $S = 1$, for a given order of approximation p and associated Gaussian quadrature weights and nodes $\{(\omega_i, x_i); i = 1, \dots, p\}$, we can approximate the expectations at time t by solving p perfect foresight models with shocks in $t + 1$ defined by the Gaussian quadrature nodes and averaging the p simulated paths for the endogenous variables with weights given by the Gaussian quadrature weights. More generally, if $S > 1$ we define paths for the future shocks. The figure 7 illustrates the paths of future innovations to be considered if $S = p = 3$. The most likely sequence of innovation is the central path where the innovations remain null in the future. Again the number of paths explode exponentially when S grows. If we have one scalar innovation in the model, and if we use an order p Gaussian approximation, the number of paths is p^S . If we have more than one innovation ($n_s > 1$), the total number of paths is $p^{S n_s}$. We approximate the expectations at time t by solving $p^{S n_s}$ perfect foresight models with these sequences of future innovations and averaging the $p^{S n_s}$ simulated paths for the endogenous variables with the probability of each path. Clearly, this approach suffers from the well known *Curse of Dimensionality*. Note however that the proposed computation of the expectation is an embarrassingly parallel problem¹, meaning that it is quite simple to distribute the evaluation of the expected terms on multiple threads.

The stochastic extended path approach is more formally described in algorithm 2. In this case we assume that the innovations are normally distributed. For another parametric assumption about the distribution of the innovations we may have to adapt the Gaussian quadrature rule.

Note that an alternative implementation, which can be done in a Dynare mod file (see [Adjemian et al. \(2011\)](#)), would be to build an extended model consisting in the replication of the original model for different paths of the future innovations (treated as parameters) and a set of equations for averaging the replicated endogenous variables. With this extended model, the SEP approach can be implemented using algorithm 1 instead of algorithm 2.

¹It is true that parallelization does not solve the *Curse of Dimensionality* issue, but in practice it can considerably reduce the burden cost of evaluating these integrals.

Algorithm 2 Stochastic extended path algorithm

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1. $H \leftarrow$ Set the deterministic horizon of the perfect foresight models.
 2. $S \leftarrow$ Set the stochastic horizon of the SEP solver.
 3. $p \leftarrow$ Set the order of approximation of the (Hermite) Gaussian quadrature.
 4. $\{(\epsilon_i, \omega_i); i = 1, \dots, p\} \leftarrow$ Compute the weights and nodes of the Gaussian quadrature.
 5. $\{(\epsilon_{1,j}, \dots, \epsilon_{S,j}, \omega_j); j = 1, \dots, p^{S_{ns}}\} \leftarrow$ Enumerate all the future paths for the innovations.
 6. $\{(\epsilon_{1,j}, \dots, \epsilon_{S,j}, \omega_j^*); j = 1, \dots, p^{S_{ns}}\} \leftarrow$ Normalize the weights so that $\sum_{j=1}^{p^{S_{ns}}} \omega_j^* = 1$.
 7. $\{(u_{1,j}, \dots, u_{S,j}, \omega_j^*); j = 1, \dots, p^{S_{ns}}\} \leftarrow$ Normalize the shocks: $u_{s,j} = \Sigma^{-\frac{1}{2}} \epsilon_{s,j} / \sqrt{2}$.
 8. $(s_1, x_1) \leftarrow$ Choose an initial condition for the state variables.
 9. **for** $t = 1$ to T **do**
 10. $y = 0 \leftarrow$ Initialize the time t endogenous non predetermined variables.
 11. $x_+ = 0 \leftarrow$ Initialize the time $t + 1$ endogenous state variables.
 12. **for** $j = 1$ to $p^{S_{ns}}$ **do**
 13. $(y_j, x_{+,j}) \leftarrow$ Solve a PF model with $y_{t+H} = y^*$ and future shocks $u_{1,j}, \dots, u_{S,j}$.
 14. $(y, x_+) = (y, x_+) + \omega_j^* \times (y_j, x_{+,j}) \leftarrow$ Update the endogenous variables.
 15. **end for**
 16. $(y_t, x_{t+1}) = (y, x_+) \leftarrow$ Set the non predetermined and state variables.
 17. $u \leftarrow$ Draw from a multivariate Gaussian distribution with variance Σ .
 18. $s_{t+1} \leftarrow Q(s_t, u)$
 19. **end for**
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3. ILLUSTRATION

We start with a simple nonlinear asset pricing model, proposed by Burnside (1998), for which there exist a closed-form solution. It is an endowment economy where a single perishable consumption good produced by a single *tree*. A representative household can hold equity to transfer consumption from one period to the next. The household's inter temporal utility is given by

$$E_t \left\{ \sum_{\tau=0}^{\infty} \beta^{t-\tau} \frac{c_{t+\tau}^\theta}{\theta} \right\} \quad \text{with } \theta \in (-\infty, 0) \cup (0, 1]$$

where c_t represents consumption in period t .

The budget constraint is given by

$$p_t e_{t+1} + c_t = (p_t + d_t) e_t$$

where p_t is the price of one unit of equity. e_t indicates the number of units of equity detained by the household at the beginning of period t and d_t is the dividend paid for one unit of equity. Dividends d_t are growing at rate x_t :

$$\begin{aligned} d_t &= \exp(x_t) d_{t-1} \\ x_t &= (1 - \rho)\bar{x} + \rho x_{t-1} + \epsilon_t \end{aligned}$$

The dynamics of this economy can be summarized by

$$\begin{aligned} y_t &= \beta E_t \{ \exp(\theta x_{t+1}) (1 + y_{t+1}) \} \\ x_t &= (1 - \rho)\bar{x} + \rho x_{t-1} + \epsilon_t \end{aligned}$$

where $y_t = p_t/d_t$ is the price-dividend ratio.

It is easy to show that y_t can be written as the current value of future dividends growth rates:

$$\begin{aligned} y_t &= E_t \left\{ \sum_{i=1}^{\infty} \beta^i \exp \left(\sum_{j=1}^i \theta x_{t+j} \right) \right\} \\ &= E_t \left\{ \sum_{i=1}^{\infty} \beta^i \exp \left(\theta \sum_{j=1}^i \bar{x} + \rho^i \hat{x}_t + \sum_{\ell=1}^i \rho^{j-\ell} \epsilon_{t+\ell} \right) \right\} \end{aligned}$$

with $\hat{x}_t = x_t - \bar{x}$.

Using formulas for the distribution of the log-normal random variable, Burnside (1998) shows that the closed form solution is

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp(a_i + b_i \hat{x}_t)$$

where

$$a_i = \theta \bar{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2} \right)$$

and

$$b_i = \frac{\theta \rho (1-\rho^i)}{1-\rho}$$

3.1. The extended path approach. In the extended path approach, one sets future shocks to their expected value, $E\{\epsilon_{t+\ell}\} = 0$, $\ell = 1, \dots, \infty$. The corresponding solution is given by

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp(a_i + b_i \hat{x}_t)$$

where

$$a_i = \theta \bar{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2} \right)$$

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where

$$a_i = \theta \bar{x} i$$

and

$$b_i = \frac{\theta \rho (1-\rho^i)}{1-\rho}.$$

3.2. **Numerical simulation.** Consider the following calibration:

$$\begin{aligned}\bar{x} &= 0.0179, \\ \rho &= -0.139, \\ \theta &= -1.5, \\ \beta &= 0.95, \\ \sigma &= 0.0348.\end{aligned}$$

Given the particular nature of the model, it is possible to compute the deterministic steady state exactly:

$$\begin{aligned}\bar{y} &= \sum_{i=1}^{\infty} \beta^i e^{\theta \bar{x} i} \\ &= \frac{\beta e^{\theta \bar{x}}}{1 - \beta e^{\theta \bar{x}}} \\ &= 12.3035\end{aligned}$$

It is also possible to compute the risky steady state, defined as the fix point in absence of shock this period:

$$\tilde{y} = \sum_{i=1}^{\infty} \beta^i \exp\left(\theta \bar{x} i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2}\right)\right) = 12.4812$$

3.3. **Comparing expended path and closed-form solution.** We can then compute the difference between expended path approximation, \hat{y}_t and the closed-form solution, y_t .

We use 800 terms to approximate the infinite summation and run simulations over 30000 periods. We report both the maximum and minimum difference between the exact solution and the extended path approximation:

$$\begin{aligned}\min(y_t - \hat{y}_t) &= 0.1726 \\ \max(y_t - \hat{y}_t) &= 0.1820\end{aligned}$$

The exact solutions takes into account the non-linear effect of future volatility and is systematically higher than the extended path approximation that is computed as if future shocks were always zero.

The effect of future volatility isn't trivial and can be apprehended by looking at the difference between the deterministic and the risky steady state. The relative difference is equal to

$$\frac{\tilde{y} - \bar{y}}{\bar{y}} = 1.44\%.$$

On the other hand, the small difference between minimum and maximum differences reported above suggests that the effect of future volatility doesn't depend much on the state of the economy.

3.4. Stochastic extended path. A k -order stochastic extended path approach computes the conditional expectation taking into accounts the shocks over the next k periods.

The closed-form formula is

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp(a_i + b_i \hat{x}_t)$$

where

$$a_i = \theta \bar{x} i + \begin{cases} \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(i - 2\rho \frac{1-\rho^i}{1-\rho} + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2} \right) & \text{for } i \leq k \\ \frac{\theta^2 \sigma^2}{2(1-\rho)^2} \left(k - 2\rho \frac{\rho^{i-k} - \rho^i}{1-\rho} + \rho^2 \frac{\rho^{2(i-k)} - \rho^{2i}}{1-\rho^2} \right) & \text{for } i > k \end{cases}$$

and

$$b_i = \frac{\theta \rho (1 - \rho^i)}{1 - \rho}$$

3.5. Quantitative evaluation. We want to examine to which extent, the stochastic extended path approach is able to take into account the effect of future volatility on today's decision. To do so, we use stochastic extended path at different orders to compute the risky steady state. In this model, the risky steady state is simply computed by setting to zero the shock of the current period and considering only the distribution of future shocks over the k next periods. As mentioned above, for the given calibration, the deterministic steady state is equal to 12.3035 and the risky steady state, 12.4812.

In Table 1, we report the percentage of the difference between the theoretical value of the risk steady state and the deterministic steady state that is actually taken into account by a stochastic extended path approach of order k .

k	Percentage
1	7.4%
2	14.3%
9	50.0%
30	90.1%
60	99.0%

TABLE 1. $(\tilde{y} - \bar{y})/\bar{y}$, in percentage

It is obvious that a stochastic extended path approach of large order is required to give full account of the effect of future volatility on today's decisions. In the last section of the paper, we propose a hybrid approach using ideas from the perturbation method to better take into account the effect of future volatility.

4. DESCRIPTION OF THE MODELS

This section describes the basic DSGE models used as benchmarks in this paper. We consider two Real Business Cycle models with a CES production function in order to control the degree of non linearity and assess the sensitivity of the accuracy errors of the (Stochastic) Extended Path approach with respect to the non linearity. Assuming that the investment may be irreversible (model # 2) allows us to evaluate the consequences of the occasionally binding constraints in terms of accuracy.

4.1. **Model #1 (Standard RBC model).** The social planner problem is as follows:

$$\begin{aligned}
 \max_{\{c_{t+j}, l_{t+j}, k_{t+1+j}\}_{j=0}^{\infty}} \quad & \mathcal{W}_t = \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) \\
 \text{s.t.} \quad & \\
 \text{(SP}_1\text{-1)} \quad & y_t = c_t + i_t \\
 & y_t = A_t f(k_t, l_t) \\
 & k_{t+1} = i_t + (1 - \delta)k_t \\
 & A_t = A^* e^{a_t - \frac{1}{2} \frac{\sigma_\varepsilon^2}{1 - \rho^2}} \\
 & a_t = \rho a_{t-1} + \varepsilon_t
 \end{aligned}$$

with the following specifications:

$$\text{(SP-2)} \quad u(c_t, l_t) = \frac{(c_t^\theta (1 - l_t)^{1-\theta})^{1-\tau}}{1 - \tau}$$

and

$$\text{(SP-3)} \quad f(k_t, l_t) = \left(\alpha k_t^\psi + (1 - \alpha) l_t^\psi \right)^{\frac{1}{\psi}}.$$

where ε_t is a Gaussian white noise with zero mean and variance σ_ε^2 . The partial derivatives are

$$\begin{aligned}
 u_c(c_t, l_t) &= \theta \frac{(c_t^\theta (1 - l_t)^{1-\theta})^{1-\tau}}{c_t} \\
 u_l(c_t, l_t) &= -(1 - \theta) \frac{(c_t^\theta (1 - l_t)^{1-\theta})^{1-\tau}}{l_t} \\
 f_k(k_t, l_t) &= \alpha k_t^{\psi-1} \left(\alpha k_t^\psi + (1 - \alpha) l_t^\psi \right)^{\frac{1}{\psi}-1} \\
 f_l(k_t, l_t) &= (1 - \alpha) l_t^{\psi-1} \left(\alpha k_t^\psi + (1 - \alpha) l_t^\psi \right)^{\frac{1}{\psi}-1}.
 \end{aligned}$$

The first order conditions are given by:

$$(2a) \quad u_c(c_t, l_t) - \beta \mathbb{E}_t \left[u_c(c_{t+1}, l_{t+1}) \left(A_{t+1} f_k(k_{t+1}, l_{t+1}) + 1 - \delta \right) \right] = 0$$

$$(2b) \quad - \frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} - A_t f_l(k_t, l_t) = 0$$

$$(2c) \quad c_t + k_{t+1} - A_t f(k_t, l_t) - (1 - \delta)k_t = 0$$

4.2. Model #2 (RBC model with irreversible investment). We now suppose that the social planner is constrained to positive investment paths. We restate its problem as:

$$(SP_2-1) \quad \begin{aligned} \max_{\{c_{t+j}, l_{t+j}, k_{t+1+j}\}_{j=0}^{\infty}} \quad & \mathcal{W}_t = \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) \\ \text{s.t.} \quad & y_t = c_t + i_t \\ & y_t = A_t f(k_t, l_t) \\ & k_{t+1} = i_t + (1 - \delta)k_t \\ & i_t \geq 0 \\ & A_t = A^* e^{a_t - \frac{1}{2} \frac{\sigma_\varepsilon^2}{1 - \rho^2}} \\ & a_t = \rho a_{t-1} + \varepsilon_t \end{aligned}$$

where the technology (f) and the preferences (u) are respectively defined in [SP-3](#) and [SP-2](#). The first order conditions are given by:

$$(3a) \quad u_c(c_t, l_t) - \mu_t = \beta \mathbb{E}_t \left[u_c(c_{t+1}, l_{t+1}) \left(A_{t+1} f_k(k_{t+1}, l_{t+1}) + 1 - \delta \right) - \mu_{t+1} (1 - \delta) \right]$$

$$(3b) \quad - \frac{u_l(c_t, l_t)}{u_c(c_t, l_t)} - A_t f_l(k_t, l_t) = 0$$

$$(3c) \quad c_t + k_{t+1} - A_t f(k_t, l_t) - (1 - \delta)k_t = 0$$

$$(3d) \quad \mu_t (k_{t+1} - (1 - \delta)k_t) = 0$$

where μ_t is the Lagrange multiplier associated to the positiveness constraint on investment.

5. NUMERICAL ILLUSTRATION AND ACCURACY CHECKS

Figure 1 plots the evolution of investment in the RBC model with irreversible investment with stochastic extended path at different orders. We use a 3-node quadrature formula. It illustrates that, close to non-negativity boundary, the more one takes into account future uncertainty the bigger the precaution in the form of a larger investment. Note that the Figure is a window out

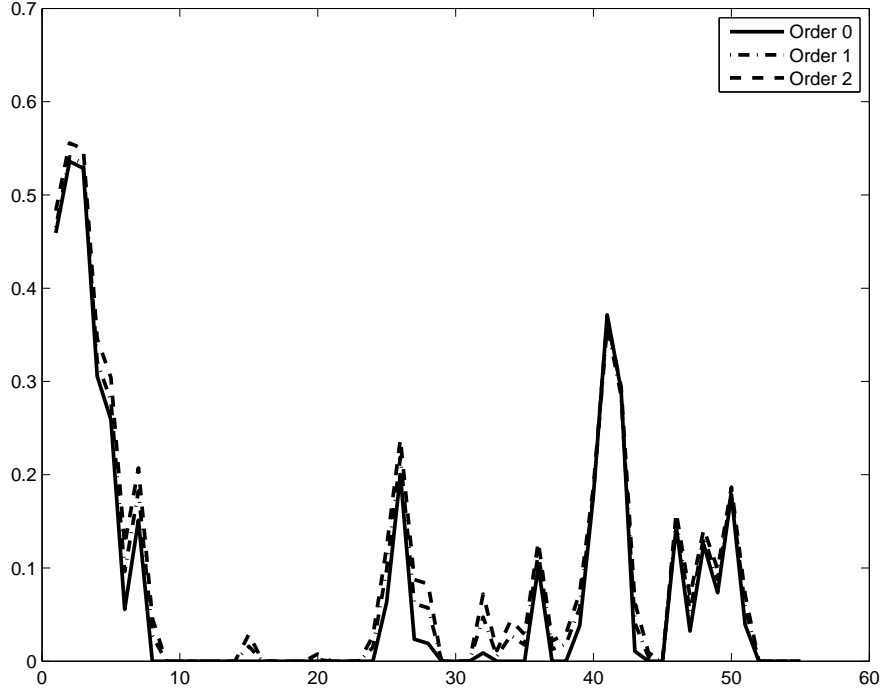


FIGURE 1. Simulated trajectory of investment for different approximation schemes

of a longer simulation and that the initial differences in the different trajectories result from previous events, not shown in the Figure.

Using the (S)EP approach to solve (1), we can, in principle, perfectly control the accuracy of the solution with respect to the deterministic equations (1a), (1c) and (1d). Consequently, we only have to check the accuracy of the solution with respect to the Euler type equations (1b), which can be rewritten as a multivariate integral:

$$\mathcal{R}(x, s) \triangleq \int_{\Lambda} F(y, x, s, H(y_+(u), x_+(u), Q(s, u))) dP(u) = 0 \quad \forall (x, s) \in \Xi \subseteq \mathbb{R}_+^{n_x + n_s}$$

where $\Lambda \subseteq \mathbb{R}^{n_s}$ and $(y, y_+(u), x_+(u))$ is provided by the stochastic extended path algorithm 2 given initial conditions (x, s) . Provided that the number of nodes used to approximate the integral defining the Euler residual $\mathcal{R}(x, s)$ is the same that the number of nodes (p) used in the SEP algorithm 2, this measure of the Euler residual is zero by construction (for $S = 1$). Moreover, because the output of the (S)EP algorithms are time series for the endogenous variables rather than policy rules and transition functions, it is more natural to compare the different algorithms following the [Den Haan and Marcet \(1994\)](#) approach.

Given simulated time series for the endogenous variables, $\{\bar{x}_t, \bar{y}_t, \bar{s}_t, \bar{u}_t\}_{t=1}^T$ provided by algorithms 1 or 2, we define the expectation errors as:

$$\bar{e}_{t+1} = F(\bar{y}_t, \bar{x}_t, \bar{s}_t, H(\bar{y}_{t+1}, \bar{x}_{t+1}, \bar{s}_{t+1}))$$

If the model is correctly solved, these residuals must be orthogonal to any non linear functions of variables appearing in the information set. More formally, the rational expectation moment condition must be satisfied:

$$\mathbb{E}[e_{t+1} \otimes h(x_t, s_t)] = 0$$

for any function h of the state variables. In practice we prefer the solution method yielding the sample rational expectation condition moment closer to zero. Let

$$B_T = T^{-1} \sum_{t=1}^T \bar{e}_{t+1} \otimes h(\bar{x}_t, \bar{s}_t)$$

be the sample rational expectation condition moment, and define the following statistic

$$\mathcal{S}_T = TB_T A_T^{-1} B_T$$

where the weighting matrix A_t is a consistent estimator of the long run variance of $e_{t+1} \otimes h(x_t, s_t)$, that is a consistent estimator of:

$$S^2 = \sum_{i=-\infty}^{\infty} \mathbb{E} \left[(e_{t+1} \otimes h(x_t, s_t)) (e_{t+1} \otimes h(x_t, s_t))' \right]$$

Den Haan and Marcet (1994) establish that, if the simulated time series solve the rational expectation model and if the long run variance S^2 is finite and invertible, we must have in the limit :

$$\mathcal{S}_T \xrightarrow[T \rightarrow \infty]{} \chi^2(k)$$

where k is the number of moment conditions (*i.e.* the number of elements in the vector $e_{t+1} \otimes h(x_t, s_t)$). In the sequel, we use this asymptotic result to compute p-values and compare the different solution strategies.

6. EXTENSIONS

The previous experiments point to two problems: the number of nodes increases very rapidly as the order of the stochastic extended path approach increases and it would take very high orders to fully account for the effect of future volatility in models such as Burnside (1998). We propose two extensions to address these problems.

The first one suggests to use sparser trees and less nodes to evaluate numerically the conditional expectation. The second uses a hybrid approach including results from the perturbation method to account for the effect of future volatility.

6.1. Sparse trees. In the basic method described above, all branches of the tree have the same length and switch back to the non-stochastic extended path method at the same period. But

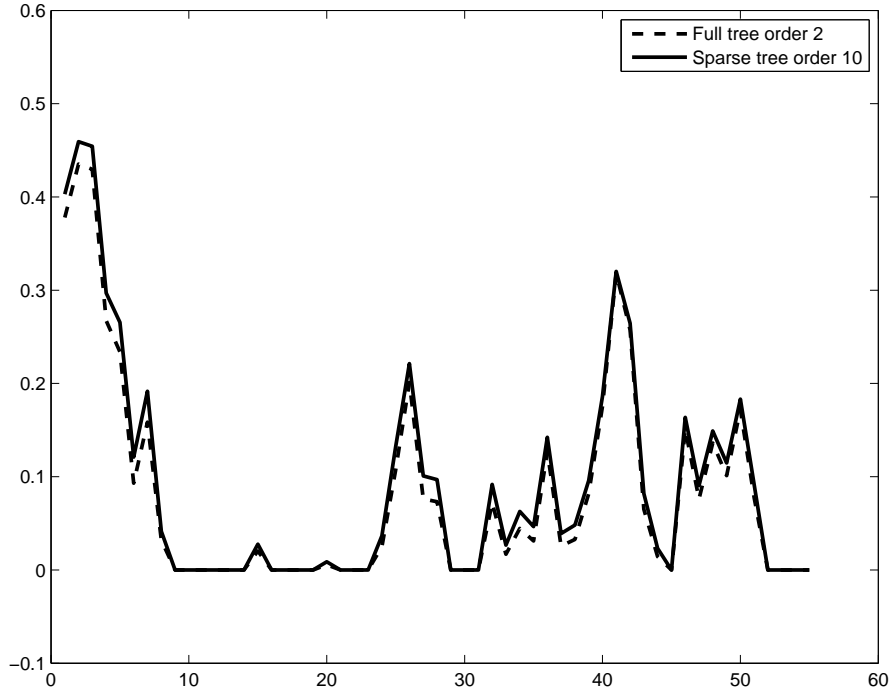


FIGURE 2. Simulated trajectory of investment for different approximation schemes

this implies considering nodes with very little weight in the computation of the conditional expectation. For example, in Figure 7, the integration weights are as follows:

$$\omega_1 = 0.1667,$$

$$\omega_2 = 0.6667,$$

$$\omega_3 = 0.1667.$$

This implies that the weight in the computation of the conditional expectation at period t , of the very top node is $0.1667^3 = 0.0046$, while the weight of the middle node, is $0.6667^3 = 0.2963$. The idea that we have been experimenting with is to extend only the branch with the largest weight, corresponding to the central node, and to switch back immediately to the non-stochastic extended path approach for the nodes with smaller weights. The resulting tree for 3 nodes, where the longest branch last 3 periods, is displayed in Figure 7.

Figure 2 compares the simulation of the RBC model with irreversible investment with a full tree at order 2 and a sparse tree at order 10. A higher order permits to take into account the asymmetric effect of possibly hitting the constraint further in the future. The den Haan and Marcet statistic² also confirms that the simulation with a sparse tree of order 10 generates a one

²We use 10000 observations, discarding the first 500 to compute the den Haan and Marcet statistic for the one period ahead prediction of consumption, using only the constant as instrument

period ahead prediction error for consumption that is less predictable than for the simulation with a full tree of order 2. The den Haan Marcet statistic is 38.9 for the sparse tree approach of order 10, but 48.8 for the full tree of order 2.

6.2. A hybrid approach. As illustrated in the previous section, it is necessary to factor in the expectations for a large number of periods forward (the order of stochastic extended path) to obtain an accurate figure of the effects of future volatility. However, even a local approximation with a Taylor expansion of low order provides better information on this effect of future volatility. This suggests to combine the two approaches. In the perturbation approach, the effect of future volatility is summarized by the contribution of σ , the stochastic scale of the model. We suggest therefore to correct the deterministic simulation conducted for the periods beyond the order of the stochastic extended path approach by adding the contribution of the stochastic scale provided by the perturbation approach.

For example, mixing a perturbation approach of order 2 and a stochastic extended path approach of order 1, the equations for node i would be:

$$(4a) \quad s_t = Q(s_{t-1}, u_t)$$

$$(4b) \quad F(y_t, x_t, s_t, E_{t+1}) = 0$$

$$(4c) \quad G(y_t, x_{t+1}, x_t, s_t) = 0$$

$$(4d) \quad \mathcal{E}_t = H(y_t + g_{\sigma\sigma}/2, x_t, 0)$$

where the value of y_t is corrected in the fourth equation.

When using this approach to simulate Burnside (1998) model, with a stochastic extended path algorithm of order 2, the hybrid approach delivers a spectacular improvement in accuracy.

As before, we compute the difference between the stochastic extended path approximation of order 2, the hybrid approach of order 2 and the closed-form solution, y_t . We use 800 terms to approximate the infinite summation and run simulations over 30000 periods. We report both the maximum and minimum difference of the two approaches with the exact solution:

	Stochastic extend path	Hybrid stochastic extend path
maximum difference	0.1607	0.0021
minimum difference	0.1513	0.0019

On the other hand, for the RBC model with irreversible investment, the hybrid approach makes little difference. The maximum absolute difference between a second order stochastic extended path or a second order hybrid approach is only 0.0033.

7. CONCLUSION

This paper presents the stochastic extended path approach to simulate non linear models while partially taking into account the systematic effect of future volatility on today decision. The method is illustrated and its properties analyzed through two models. The first one, by Burnside (2008), possess a closed form solution that permits an unambiguous measure of accuracy. The second one, a RBC model with irreversible investment, illustrates the behavior of the method when the model includes an occasionally binding constraints.

Those experiments show that is necessary to include a large number of periods for the computation of the conditional expectation at the heart of this type of models in order to adequately take into account the effect of future volatility. This can be prohibitively expensive to compute in practice.

In the last section of the paper, we present two extensions: the use of sparse tree and a hybrid method using a perturbation approach to deal with the effect of future volatility. Both extensions improve the performance of the stochastic extended path approach.

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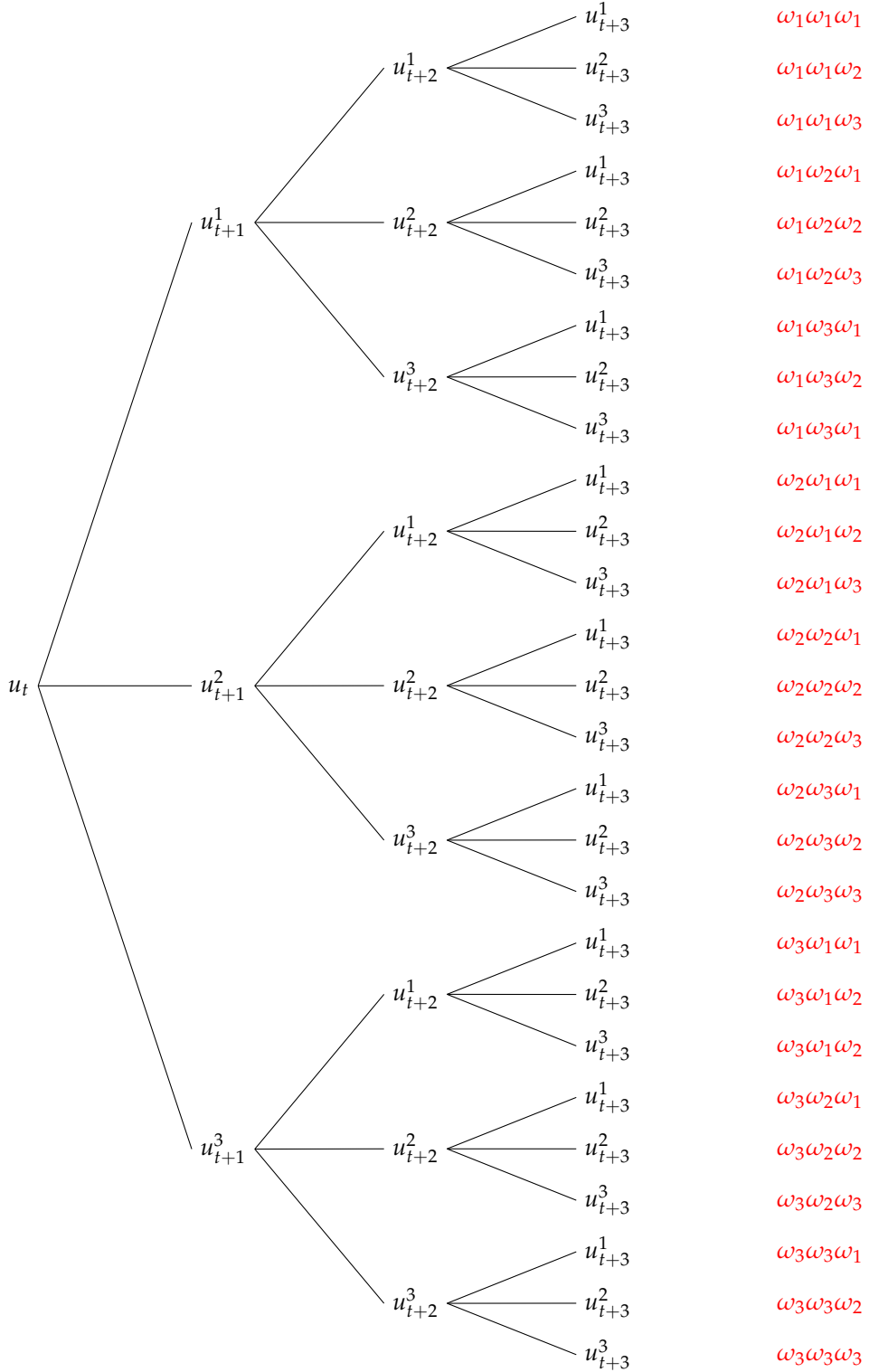


FIGURE 3. **Paths of future innovations.** Illustration with a scalar innovation, u , $S = 3$, and $p = 3$ Hermite Gaussian quadrature nodes $x^1 = -\frac{\sqrt{6}}{2}$, $x^2 = 0$, $x^3 = \frac{\sqrt{6}}{2}$, and the associated quadrature weights $\omega_1 = \frac{\sqrt{\pi}}{6}$, $\omega_2 = \frac{2\sqrt{\pi}}{3}$, $\omega_3 = \frac{\sqrt{\pi}}{6}$. By construction we have that $\sum_{i,j,k=1}^p \omega_i\omega_j\omega_k = \pi^{\frac{3}{2}}$. Up to the constant of integration $\pi^{\frac{3}{2}}$, the cumulated weight $\omega_i\omega_j\omega_k$ has to be interpreted as the probability of a particular sequence of future innovations. The central path, *ie* the sequence $\{u_{t+1}^2, u_{t+2}^2, u_{t+3}^2\}$, is the most likely, its probability is $\omega_2\omega_2\omega_2\pi^{-\frac{3}{2}} = 8/27$.

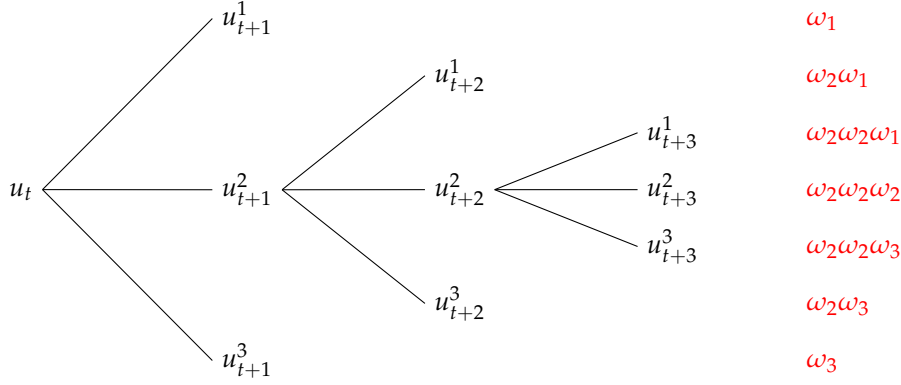


FIGURE 4. **Paths of future innovations in a sparse tree.** Illustration with a scalar innovation, u , $S = 3$, and $p = 3$

APPENDIX A. RBC MODELS

A.1. **Model #1.** The dynamic of the model is characterized by the following equations:

$$(A.1a) \quad \begin{aligned} & \left[c_t^\theta (1-l_t)^{1-\theta} \right]^{-\tau} \theta c_t^{\theta-1} (1-l_t)^{1-\theta} \\ & - \beta \mathbb{E}_t \left[\left[c_{t+1}^\theta (1-l_{t+1})^{1-\theta} \right]^{-\tau} \theta c_{t+1}^{\theta-1} (1-l_{t+1})^{1-\theta} \right. \\ & \quad \left. \times \left\{ \alpha \left[\alpha + (1-\alpha) \left(\frac{k_{t+1}}{l_{t+1}} \right)^{-\psi} \right]^{\frac{1-\psi}{\psi}} A_{t+1} + 1 - \delta \right\} \right] = 0 \end{aligned}$$

$$(A.1b) \quad \frac{1-\theta}{\theta} \frac{c_t}{1-l_t} - (1-\alpha) A_t \left[\alpha \left(\frac{k_t}{l_t} \right)^\psi + 1 - \alpha \right]^{\frac{1-\psi}{\psi}} = 0$$

$$(A.1c) \quad c_t + k_{t+1} - A_t \left[\alpha k_t^\psi + (1-\alpha) l_t^\psi \right]^{\frac{1}{\psi}} - (1-\delta) k_t = 0$$

and the law of motion for efficiency.

A.2. **Model #2.** The dynamic of the model is characterized by the following equations:

$$(A.2a) \quad \begin{aligned} & \left[c_t^\theta (1-l_t)^{1-\theta} \right]^{-\tau} \theta c_t^{\theta-1} (1-l_t)^{1-\theta} - \mu_t \\ & - \beta \mathbb{E}_t \left[\left[c_{t+1}^\theta (1-l_{t+1})^{1-\theta} \right]^{-\tau} \theta c_{t+1}^{\theta-1} (1-l_{t+1})^{1-\theta} \right. \\ & \quad \left. \times \left\{ \alpha \left[\alpha + (1-\alpha) \left(\frac{k_{t+1}}{l_{t+1}} \right)^{-\psi} \right]^{\frac{1-\psi}{\psi}} A_{t+1} + 1 - \delta \right\} - \mu_{t+1} (1-\delta) \right] = 0 \end{aligned}$$

$$(A.2b) \quad \frac{1-\theta}{\theta} \frac{c_t}{1-l_t} - (1-\alpha)A_t \left[\alpha \left(\frac{k_t}{l_t} \right)^\psi + 1 - \alpha \right]^{\frac{1-\psi}{\psi}} = 0$$

$$(A.2c) \quad c_t + k_{t+1} - A_t \left[\alpha k_t^\psi + (1-\alpha)l_t^\psi \right]^{\frac{1}{\psi}} - (1-\delta)k_t = 0$$

$$(A.2d) \quad \mu_t (k_{t+1} - (1-\delta)k_t) = 0$$

and the law of motion for efficiency.

APPENDIX B. STEADY STATE OF THE MODEL

To obtain the analytical expression of the steady state, we define some the ratios of the endogenous variables as functions of the deep parameters. From the Euler equation we have:

$$\frac{y^*}{k^*} = \left(\frac{\beta^{-1} - 1 + \delta}{\alpha} \right)^{\frac{1}{1-\psi}}$$

From the resource constraint:

$$\frac{c^*}{k^*} = \frac{y^*}{k^*} - \delta$$

From the definition of the production function, we define the steady state level of labor and physical capital average productivity:

$$\begin{aligned} \frac{y^*}{l^*} &= A^* \left[\alpha + (1-\alpha) \left(\frac{l^*}{k^*} \right)^\psi \right] \\ \frac{y^*}{k^*} &= A^* \left[\alpha \left(\frac{k^*}{l^*} \right)^\psi + (1-\alpha) \right] \end{aligned}$$

Substituting the expression of y^*/k^* in the last equation, we obtain the following intermediary result:

$$\frac{l^*}{k^*} = \left[\left(\frac{\beta^{-1} - 1 + \delta}{\alpha} \right)^{\frac{\psi}{1-\psi}} A^{*\psi} - \alpha \right]^{\frac{1}{\psi}} (1-\alpha)^{-\frac{1}{\psi}}$$

From the consumption – leisure trade off condition, we have:

$$(\star) \quad c^* = \gamma_1 (1 - l^*)$$

with

$$\gamma_1 = \theta \frac{1-\alpha}{1-\theta} \left[\frac{y^*/k^*}{l^*/k^*} \right]^{1-\psi}$$

But we also have:

$$\frac{c^*}{l^*} = \frac{c^*}{k^*} \frac{k^*}{l^*}$$

and by substitution:

$$(\star\star) \quad c^* = \gamma_2 l^*$$

with

$$\gamma_2 = \left[\left(\frac{\beta^{-1} - 1 + \delta}{\alpha} \right)^{\frac{1}{1-\psi}} - \delta \right] \left[\left(\frac{\beta^{-1} - 1 + \delta}{\alpha} \right)^{\frac{\psi}{1-\psi}} A^{*\psi} - \alpha \right]^{-\frac{1}{\psi}} (1 - \alpha)^{\frac{1}{\psi}}$$

By Equating (*) and (**) we obtain the steady state level of labor:

$$l^* = \left(1 + \frac{\gamma_2}{\gamma_1} \right)^{-1}$$

The steady state levels of the remaining endogenous variables are easily deduced from this expression and the steady state ratios previously defined. From these results we can compute the steady state level of the share of physical capital revenues:

$$s(k^*, l^*) = \frac{\alpha^{\frac{1}{1-\psi}} A^{*\psi}}{(\beta^{-1} - 1 + \delta)^{\frac{\psi}{1-\psi}}}$$

which can be used for the calibration of the model. Note that $1/1-\psi$ is the elasticity of substitution between physical capital and labor. In the Cobb–Douglas case ($\psi = 0$), we have $s(k^*, l^*) = \alpha$.

GAINS, UNIVERSITÉ DU MAINE AND CEPREMAP

E-mail address: stephane.adjemian@ens.fr

URL: www.dynare.org/stepan

BANQUE DE FRANCE AND CEPREMAP.

E-mail address: michel.juillard@mjuif.fr