Perturbation methods for DSGE models

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March, 2016

Introduction

- In this chapter we show how to solve DSGE models using perturbation technics.
- Basically, the idea is to replace the original problem by a simpler one, without loosing the properties of interest in the original model (if possible).
- This auxiliary model is obtained by perturbing the original model in the vicinity of the original model's deterministic steady state.
- We will show how we can easily solve the auxiliary model.
- It is important to understand that we do not approximate the solution of the DSGE model. We rather compute the exact solution of an approximation of the original DSGE model, hoping it provides an accurate approximation of the solution of the original DSGE model.

Outline

Perturbation approach

Square root function

- Suppose that we need to compute $\sqrt{1+\epsilon}$ for small values of ϵ...
- But that the computational burden of such an operation is very high.
- We approximate this task using a famous result from Newton:

Generalized binomial theorem

For all $(x, y) \in \mathbb{R}^2$ such that $|x/y| > 1$ and for all $r \in \mathbb{R}$ we have:

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k$$

where the binomial coefficient is defined as follows:

$$\binom{r}{k} = \frac{\prod_{i=0}^{k-1}(r-i)}{\prod_{i=0}^{k-1}(k-i)} = \frac{r^{\underline{k}}}{k!}$$

See Graham, Knuth and Patashnik (1994).

Perturbation approach

Square root function approximation

- Applying this theorem for $r = 1/2$, we find the following expression:

$$\begin{aligned}\sqrt{1+\varepsilon} &= \sum_{k=0}^{\infty} \binom{1/2}{k} \varepsilon^k \\ &= 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \frac{1}{16}\varepsilon^3 - \frac{5}{128}\varepsilon^4 + \frac{7}{256}\varepsilon^5 + \cdots\end{aligned}$$

- The power function with integer exponent is much easier to evaluate than the square root function.
- But the theorem states that we should evaluate an infinite number of power functions!
- Noting that the terms of the infinite series are rapidly converging to zero, provided $|\varepsilon| < 1$, we can truncate this expression. For instance:

$$\begin{aligned}&\sqrt{1+\varepsilon} = 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right) \\ \Rightarrow \quad &\sqrt{1+\varepsilon} \simeq 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2\end{aligned}$$

Perturbation approach

Square root function approximation error

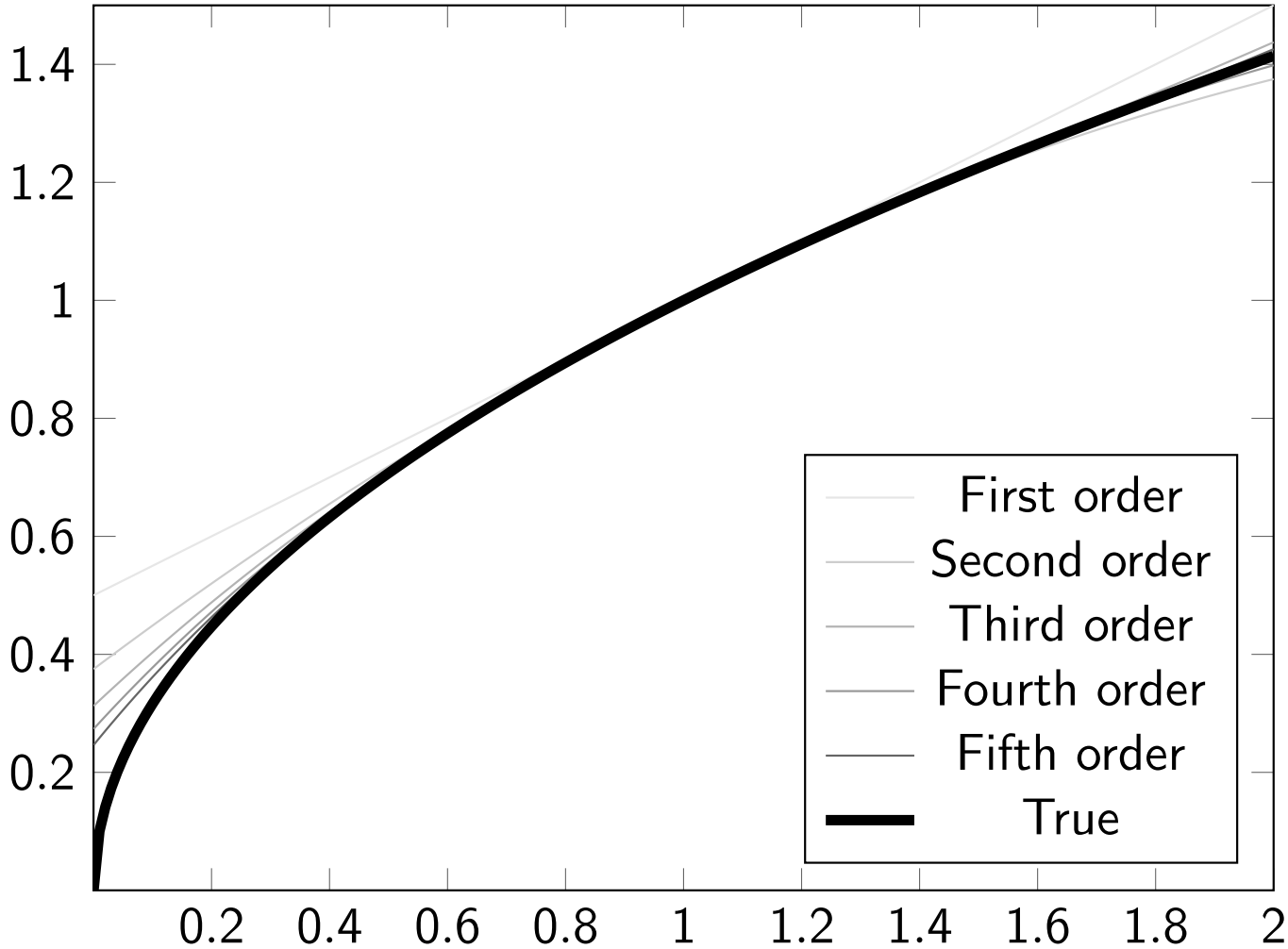
- The symbol $\mathcal{O}\left(\varepsilon^3\right)$, to be read *big 'O' of* ε *cubed*, hides the rest of the infinite series.
- This symbol means that for sufficiently small values of ε there exists a positive constant Γ *independent of* ε such that the absolute value of $\mathcal{O}\left(\varepsilon^3\right)$ is less than $\Gamma|\varepsilon|^3$.
- More generally, when we approximate a function $f(\varepsilon)$ by a truncated infinite series,

$$f(\varepsilon) = \sum_{i=0}^{p-1} c_i f^{(i)}(0)\varepsilon^i + \mathcal{O}\left(\varepsilon^p\right)$$

$\mathcal{O}\left(\varepsilon^p\right)$ means that the accuracy error does not grow faster than ε at the power p when ε is small.

Perturbation approach

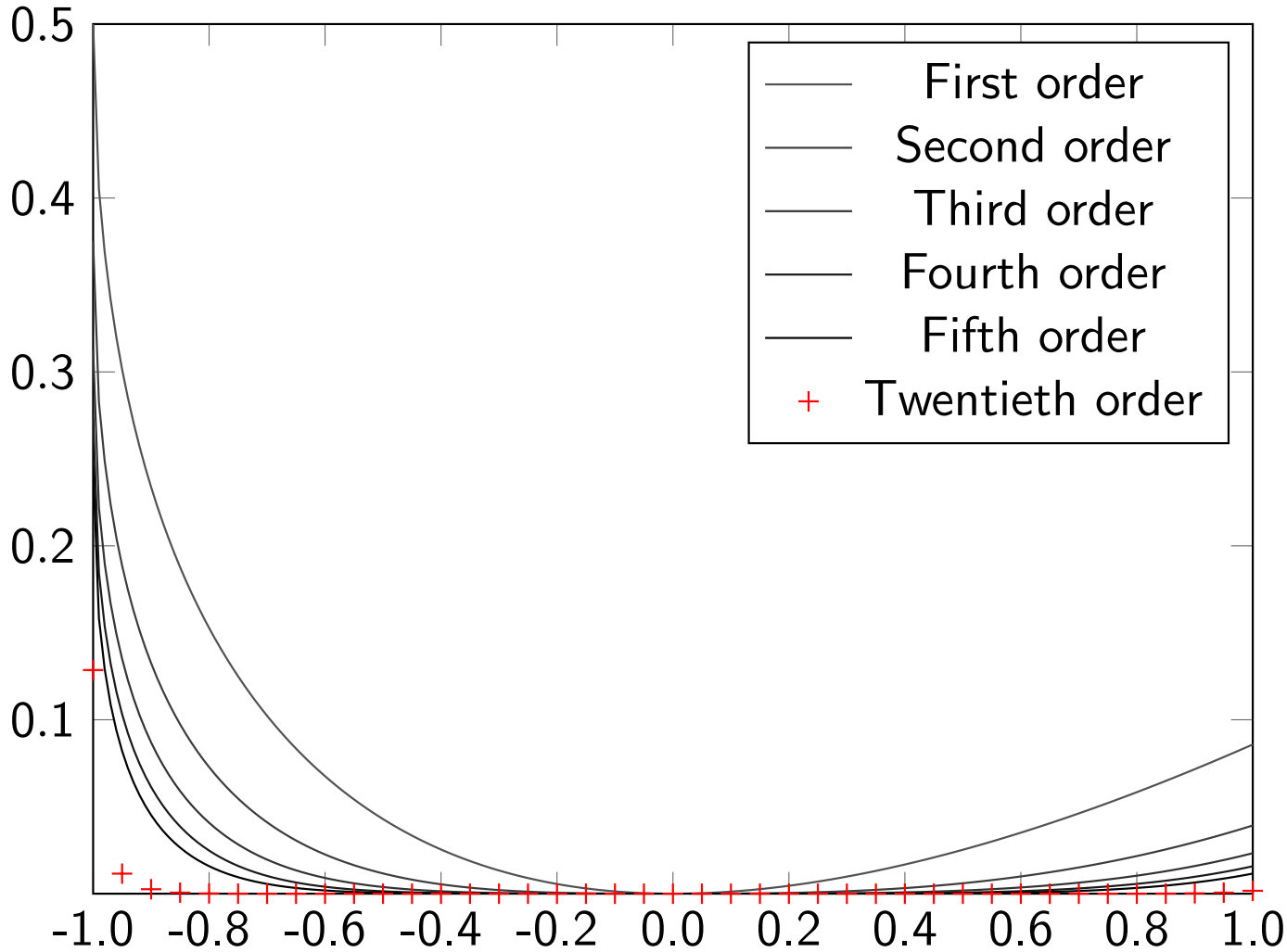
Square root function approximation error



Five approximations to $\sqrt{1+\varepsilon}$. The bold curve is the graphical representation of the true square root function between 0 and 2. The other curves represents the approximations of the square root function around $x = 1$ for ε ranging from -1 to 1.

Perturbation approach

Square root function approximation error



Approximation errors. Each curve represents the absolute value of the difference between the true function and its approximation, for different values of ε.

Perturbation approach

Square root function approximation error

- The higher is the approximation (truncation) order, the closer is the approximation to the true function.
- A striking feature is that the approximation errors are smaller for positive values of ε than for negative values.
- The square root function is much more curved at the origin (we have an infinite slope at zero) than above one.
- Obviously these approximations are not valid for any values of ε.
- The perturbations ε have to be small. But what is a small ε?
- The generalized binomial theorem assumes that ε is less than one in absolute value so that the infinite series exists.
- If $\varepsilon > 1$, the infinite series cannot exist because $\lim_{p\to\infty} \varepsilon^p = \infty$.
- In this context, a small ε is any $\varepsilon \in (-1, 1)$, we define $r = 1$ as the *radius of convergence*.
- Put differently, one can expect that the approximation will behave very poorly if $\varepsilon > 1$.
- The determination of the radius of convergence is generally not obvious (unknown in the case of DSGE models).

Perturbation approach

Square root function and its approximations (timing)

- In the following table we report the relative execution time (smaller is better) and approximation error for three approximations of $\sqrt{1+\epsilon}$ with $\epsilon = .01$.
- The time execution is relative to the direct computation of the square root.
- Polynomials (approximation order greater than one) are computed with the Horner scheme.
- Matlab code is available here.

Approx. order	Relative time	Approx. error
1	.2502	1.2438×10^{-5}
2	.5220	-6.2112×10^{-8}
3	.7947	3.8791×10^{-10}

Stochastic RBC model

Equations

As an example, consider the RBC model, where the dynamics of consumption, physical capital and productivity are given by:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{\alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} + 1 - \delta}{c_{t+1}} \right] \tag{1}$$

$$k_{t+1} = e^{a_t} k_t^{\alpha} + (1-\delta) k_t - c_t \tag{2}$$

$$a_t = \varphi a_{t-1} + \epsilon_t \tag{3}$$

- $\{\epsilon_t\} \sim iid(0, \sigma_\epsilon^2)$, usually the distribution of the innovations is Gaussian.
- $\mathbb{E}_t[X_{t+1}]$ is the expectation conditional on the information available at time t.
- The information set at time t contains the previous realizations of the endogenous variables, the contemporaneous innovations and the variables decided at time t).

Log linearization

- Suppose that we have the following recurrent equation:

$$x_t = f(x_{t-1})$$

  with the steady state $x^\star$ such that $x^\star = f(x^\star)$, which is assumed to be non zero.
- Define $\tilde{x}_t$ such that $x_t = x^\star e^{\tilde{x}_t}$, or equivalently $\tilde{x}_t = \log x_t - \log x^\star$ the percentage deviation from the steady state.
- We can rewrite the recurrent equation in terms of $\tilde{x}_t$:

$$x^\star e^{\tilde{x}_t} = f\left(x^\star e^{\tilde{x}_{t-1}}\right)$$

- A first order Taylor approximation of both sides around $\tilde{x}_t = 0$ gives:

$$x^\star + x^\star \tilde{x}_t \approx f(x^\star) + x^\star f'(x^\star)\tilde{x}_{t-1}$$

$$\Leftrightarrow \tilde{x}_t \approx f'(x^\star)\tilde{x}_{t-1}$$

Stochastic RBC model

Log linearization

- The exogenous variable a_t is already in logarithm and its law of motion is linear, we only log-linearize with respect to c_t and k_t.

Exercise 1.

Show that the log linearized version of (1)-(2) is given by:

$$\mathbb{E}_t\left[\tilde{c}_t - \tilde{c}_{t+1} + \frac{\rho+\delta}{1+\rho}\left(\tilde{a}_{t+1} - (1-\alpha)\tilde{k}_{t+1}\right)\right] = 0 \tag{4}$$

$$\tilde{k}_{t+1} = \frac{y^\star}{k^\star}a_t + \beta^{-1}\tilde{k}_t - \frac{c^\star}{k^\star}\tilde{c}_t \tag{5}$$

with $\tilde{a}_t = a_t$.

- We do not need to compute explicitly the deterministic steady state to approximate the model around the deterministic steady state! $\Rightarrow$ Steady state ratios.
- We even do not need to specify functional forms...

Stochastic RBC model

Log linearization (without explicit functions)

Exercise 2.

Suppose that the Euler and transition equations are given by:

$$u'(c_t) = \beta \mathbb{E}_t \left[u'(c_{t+1} \left(e^{a_{t+1}} f'(k_{t+1}) + 1 - \delta \right)) \right]$$
$$k_{t+1} = e^{a_t} f(k_t) + (1-\delta) k_t - c_t$$

where $y_t = e^{a_t} f(k_t)$ is the level of production, $f(k)$ is a neoclassical production function, and $u(c)$ is the instantaneous utility function. Let α be the elasticity of output with respect to capital at the steady state and γ be the absolute value of the elasticity of the marginal utility with respect to consumption at the steady state. **(1)** Characterize the steady state. **(2)** Compute the steady state ratios $c^\star/k^\star$ and $y^\star/k^\star$. **(3)** Show that the log-linearized Euler and transition equations are:

$$\mathbb{E}_t \left[\gamma \left(\tilde{c}_t - \tilde{c}_{t+1} \right) + \frac{\rho + \delta}{1 + \rho} \left(\tilde{a}_{t+1} - (1-\alpha) \tilde{k}_{t+1} \right) \right] = 0$$
$$\frac{y^\star}{k^\star} a_t + \beta^{-1} \tilde{k}_t - \frac{c^\star}{k^\star} \tilde{c}_t - \tilde{k}_{t+1} = 0$$

Stochastic RBC model

Solution of the log linearized model

- A solution is a time invariant mapping between the states (a_t and k_t) and the controls (c_t, k_{t+1}).
- If $c_t = \psi(k_t, a_t)$ is known, one can build time series for all the endogenous variables by iterating over (2)-(3).
- Except under rare occasions, it is generally not possible to obtain a closed form solution for this mapping.

Exercise 3.

Show that it is possible to solve analytically the previous RBC model if $\delta = 1$.

- If the model is linear (or linearized) one can show that the solution is linear (provided that the solution exists).
- We postulate a linear solution:

$$\begin{aligned} c_t &= \eta_{ck} k_t + \eta_{ca} a_t \\ k_{t+1} &= \eta_{kk} k_t + \eta_{ka} a_t \end{aligned} \tag{6}$$

A unique solution exists iff there exists a unique vector $(\eta_{ck}, \eta_{ca}, \eta_{kk}, \eta_{ka})$ such that (6) is consistent with (4), (5) and (3).

Stochastic RBC model

Solution of the log linearized model

Exercise 4.

Substitute (6) in (4), (5) and (3) and show that the reduced form parameters must satisfy:

$$\begin{cases} \eta_{ck} &= \frac{k^\star}{c^\star}\left(\beta^{-1} - \eta_{kk}\right) \\ \eta_{ca} &= \frac{y^\star}{c^\star} - \frac{k^\star}{c^\star}\eta_{ka} \\ 0 &= \frac{k^\star}{c^\star}\left(\beta^{-1} - \eta_{kk}\right)(1-\eta_{kk}) - (1-\alpha)\frac{\rho+\delta}{1+\rho}\eta_{kk} \\ 0 &= \left(\frac{y^\star}{c^\star} - \frac{k^\star}{c^\star}\eta_{ka}\right)(1-\varphi) - \frac{k^\star}{c^\star}\left(\beta^{-1} - \eta_{kk}\right)\eta_{ka} + \frac{\rho+\delta}{1+\rho}\left(\varphi - (1-\alpha)\eta_{ka}\right) \end{cases}$$

- The third equation is quadratic w.r.t η_{kk}. If we can identify a unique feasible real solution to this equation, then we can uniquely determine η_{ca} from the fourth equation, and (η_{ca}, η_{ck}), from the first and second equations.
- η_{kk} must solve:

$$\eta_{kk}^2 - \xi\eta_{kk} + \beta^{-1} = 0$$

with

$$\xi = 1 + \beta^{-1} + \frac{c^\star}{k^\star}(1-\alpha)\frac{\rho+\delta}{1+\rho} > 1 + \beta^{-1}$$

Stochastic RBC model

Solution of the log linearized model

Exercise 5.

Show that the previous quadratic equation admits two distinct real solutions: one between zero and one and the other greater than one.

- The second solution (greater than one) corresponds to a parametrization of the reduced form model where the dynamic of physical capital is explosive.
- We rule out explosive dynamics by selecting the first solution of the quadratic equation:

$$\eta_{kk} = \frac{\xi}{2} - \sqrt{\left(\frac{\xi}{2}\right)^2 - \beta^{-1}}$$

- In the process of solving a linear (or linearized) RE model we always have to solve a quadratic equation and to rule out unstable solutions...
- But if the number of endogenous states is greater than two, it is generally impossible to solve the linearized model analytically.

Stochastic RBC model

ARMA stochastic process

- The endogenous variables are ARMA processes.
- For instance, the output is characterized by:

$$\begin{cases} \tilde{y}_t &= a_t + \alpha \tilde{k}_t \\ \tilde{k}_t &= \eta_{kk} \tilde{k}_{t-1} + \eta_{ka} a_{t-1} \\ a_t &= \varphi a_{t-1} + \epsilon_t \end{cases}$$

- One can easily establish that:

$$\tilde{y}_t = (\eta_{kk} + \varphi)\tilde{y}_{t-1} - \eta_{kk}\varphi\tilde{y}_{t-2} + \epsilon_t - (\eta_{kk} - \alpha\eta_{ka})\epsilon_{t-1}$$

  An ARMA(2,1) stochastic process with two real roots in the autoregressive part (η_{kk} and φ).

Exercise 6.

Show that the distribution of y_t is log-normal if the innovation ϵ_t is Gaussian. Compute the expectation and variance of y_t. Compare $\mathbb{E}[y_t]$, the mode of the distribution of y_t and the deterministic steady state.

First order approximation

General problem

- Let y be a $n \times 1$ vector of endogenous variables, u is a $q \times 1$ vector of innovations (exogenous variables in Dynarelanguage).
- We consider the following type of model:

$$\mathbb{E}_t \left[f(y_{t+1}, y_t, y_{t-1}, u_t) \right] = 0$$

with:

$$\begin{aligned} u_t &= \sigma \epsilon_t \\ \mathbb{E}[\epsilon_t] &= 0 \\ \mathbb{E}[\epsilon_t \epsilon_t'] &= \Sigma_\epsilon \end{aligned}$$

where σ is a scale parameter, ϵ is a vector of auxiliary random variables.

- **Assumption** f : $\mathbb{R}^{3n+q} \to \mathbb{R}^n$ is a differentiable function in $\mathcal{C}^k$.

First order approximation

Solution

- The unknown function g collects the policy rules and transition equations:

$$y_t = g(y_{t-1}, u_t, \sigma)$$

- Then, we have:

$$\begin{aligned} y_{t+1} &= g(y_t, u_{t+1}, \sigma) \\ &= g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma) \end{aligned}$$

- so we can define:

$$F_g(y_{t-1}, u_t, u_{t+1}, \sigma) = f(g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t)$$

- And our problem can be restated as:

$$\mathbb{E}_t \left[F_g(y_{t-1}, u_t, u_{t+1}, \sigma) \right] = 0$$

- To solve the DSGE model we have to identify the unknown function g (solve a functional equation).
- To reduce the computational burden we approximate the problem.

First order approximation

Steady state

- A deterministic steady state, $y^\star$, for the model satisfies

$$f(y^\star, y^\star, y^\star, 0) = 0$$

- A model can have several steady states, but only one of them will be used for approximation.

- Furthermore, the solution function satisfies:

$$y^\star = g(y^\star, 0, 0)$$

- If the analytical steady state is available, it should be provided to Dynare.

First order approximation

Taylor approximation

- Let $\hat{y} = y_{t-1} - \bar{y}$, $u = u_t$, $u_+ = u_{t+1}$, $f_{y_+} = \frac{\partial f}{\partial y_{t+1}}$, $f_y = \frac{\partial f}{\partial y_t}$, $f_{y_-} = \frac{\partial f}{\partial y_{t-1}}$, $f_u = \frac{\partial f}{\partial u_t}$, $g_y = \frac{\partial g}{\partial y_{t-1}}$, $g_u = \frac{\partial g}{\partial u_t}$, $g_\sigma = \frac{\partial g}{\partial \sigma}$.

- Where all the derivates are evaluated at the deterministic steady state.

- With a first order Taylor expansion of F around $\bar{y}$:

$$
\begin{aligned}
0 \simeq {F_g}^{(1)}(y_-, u, u_+, \sigma) = \\
f_{y_+} \left(g_y \left(g_y \hat{y} + g_u u + g_\sigma \sigma \right) + g_u u_+ + g_\sigma \sigma \right) \\
+ f_y \left(g_y \hat{y} + g_u u + g_\sigma \sigma \right) + f_{y_-} \hat{y} + f_u u
\end{aligned}
$$

- **What has changed?** We now have three unknown "parameters" (g_y, g_u and g_σ) instead of an infinite number of parameters (function g).

First order approximation

Taylor approximation

- Taking the expectation conditional on the information at time t, we have:

$$\begin{aligned} 0 \simeq & f_{y_+} \left(g_y \left(g_y \hat{y} + g_u u + g_\sigma \sigma \right) + g_u \mathbb{E}_t [u_+] + g_\sigma \sigma \right) \\ & + f_y \left(g_y \hat{y} + g_u u + g_\sigma \sigma \right) + f_{y_-} \hat{y} + f_u u \end{aligned}$$

- Or equivalently:

$$\begin{aligned} 0 \simeq & \left(f_{y_+} g_y g_y + f_y g_y + f_{y_-} \right) \hat{y} + \left(f_{y_+} g_y g_u + f_y g_u + f_u \right) u \\ & + \left(f_{y_+} g_y g_\sigma + f_{y_+} g_\sigma + f_y g_\sigma \right) \sigma \end{aligned}$$

- This "equality" must hold for any value of $(\hat{y}, u, \sigma)$, so that the terms between parenthesis must be zero. We have three (multivariate) equations and three (multivariate) unknowns:

$$\begin{cases} 0 &= f_{y_+} g_y g_y + f_y g_y + f_{y_-} \\ 0 &= f_{y_+} g_y g_u + f_y g_u + f_u \\ 0 &= f_{y_+} g_y g_\sigma + f_{y_+} g_\sigma + f_y g_\sigma \end{cases}$$

First order approximation

Certainty equivalence

- Let us assume that g_y is known. We must have:

$$f_{y_+} g_y g_\sigma + f_{y_+} g_\sigma + f_y g_\sigma = 0$$

- Solving for g_σ, we obtain

$$g_\sigma = 0$$

- This is a manifestation of the certainty equivalence property of the first order approximation: the policy rules and transition equations do not depend on the size of the structural shocks.

- In this sense *future uncertainty* does not matter.

First order approximation

Recovering the marginal effect of contemporaneous innovations, g_u

- Let us assume again that g_y is known. We must have:

$$f_{y_+} g_y g_u + f_y g_u + f_u = 0$$

- Solving for g_u, we obtain

$$g_u = -\left(f_{y_+} g_y + f_y\right)^{-1} f_u$$

- Note that $f_{y_+} g_y + f_y$ must be a full rank matrix.
- g_u gives the marginal effect of the structural innovations on the endogenous (jumping and states) variables.
- Future uncertainty does not matter, but the *contemporaneous* innovations do affect the endogenous variables.

First order approximation

Recovering the marginal effect of the past, g_y

- We must have:

$$\left(f_{y_+} g_y g_y + f_y g_y + f_{y_-}\right) \hat{y} = 0 \quad \forall \hat{y}$$

- This is a quadratic equation, but the unknown is a matrix! It is generally impossible to solve this equation analytically as we would do for a univariate quadratic equation.

- If we interpret g_y as a lead operator, we can rewrite the equation as a second order recurrent equation:

$$f_{y_+} \hat{y}_{t+1} + f_y \hat{y}_t + f_{y_-} \hat{y}_{t-1} = 0$$

- For a given initial condition, $\hat{y}_{t-1}$, an infinity of paths $(\hat{y}_t, \hat{y}_{t+1})$ is solution of the second order recurrent equation.

⇜⇝ In the phase diagram of the RBC model (see the previous chapter), an infinity of trajectories satisfy the Euler and transition equations.

First order approximation

Recovering the marginal effect of the past, g_y

- The second order recurrent equation can be equivalently represented as a first order recurrent equation by increasing the dimension of the vector of endogenous variables, as we would rewrite an AR(2) as a VAR(1).

- We can rewrite the second order recurrent equation as a first order recurrent equation for $z_t \equiv (\hat{y}_t', \hat{y}_{t+1}')'$:

$$\begin{pmatrix} 0_n & f_{y+} \\ I_n & 0_n \end{pmatrix} \underbrace{\begin{pmatrix} \hat{y}_t \\ \hat{y}_{t+1} \end{pmatrix}}_{z_t} = \begin{pmatrix} -f_y & -f_{y-} \\ I_n & 0_n \end{pmatrix} \underbrace{\begin{pmatrix} \hat{y}_{t-1} \\ \hat{y}_t \end{pmatrix}}_{z_{t-1}}$$

- An admissible path z_t must also be such that the transitions, from $t-1$ to t or from t to $t+1$, are time invariant: *ceteris paribus* we have $\hat{y}_t = g_y \hat{y}_{t-1}$ **and** $\hat{y}_{t+1} = g_y \hat{y}_t$.

- In the sequel we examine the conditions under which g_y exists and allows to pin down a single stable trajectory for the endogenous variables.

First order approximation

Recovering the marginal effect of the past, g_y

- The unknown matrix g_y must be such that

$$\underbrace{\begin{pmatrix} 0_n & f_{y+} \\ I_n & 0_n \end{pmatrix}}_{D} \begin{pmatrix} I_n \\ g_y \end{pmatrix} g_y \hat{y} = \underbrace{\begin{pmatrix} -f_y & -f_{y-} \\ I_n & 0_n \end{pmatrix}}_{E} \begin{pmatrix} I_n \\ g_y \end{pmatrix} \hat{y}$$

- The matrix D is not necessarily invertible.

- We use a generalized Schur decomposition of matrices D and E.

First order approximation

Generalized Schur decomposition

- The real generalized Schur decomposition of the pencil $< E, D >$:

$$D = QTZ$$
$$E = QSZ$$

with T upper triangular, S quasi-upper triangular, $Q'Q = I$ and $Z'Z = I$.

- Generalized eigenvalues λ_i solves

$$\lambda_i D v_i = E v_i$$

For diagonal blocks on S of dimension 1×1:

- $T_{ii} \neq 0$: $\lambda_i = \frac{S_{ii}}{T_{ii}}$
- $T_{ii} = 0$, $S_{ii} > 0$: $\lambda = +\infty$
- $T_{ii} = 0$, $S_{ii} < 0$: $\lambda = -\infty$
- $T_{ii} = 0$, $S_{ii} = 0$: $\lambda \in \mathbb{C}$

Diagonal blocks of dimension 2×2 correspond to conjugate complex eigenvalues.

First order approximation

Recovering the marginal effect of the past, g_y

- Applying the Schur decomposition and multiplying by Q' we obtain:

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_n \\ g_y \end{pmatrix} g_y \hat{y} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_n \\ g_y \end{pmatrix} \hat{y}$$

- Matrices S and T are arranged in such a way that the stable eigenvalues come first.
- First block of lines, in S and T are for the stable eigenvalues. The rows of Z are partitioned accordingly.
- The columns of Z are partitioned consistently with I_n and g_y.

First order approximation

Recovering the marginal effect of the past, g_y

- g_y is identified by imposing the stability of the path.
- To exclude explosive trajectories, one must impose

$$Z_{21} + Z_{22} g_y = 0$$

- Or equivalently:

$$g_y = -Z_{22}^{-1} Z_{21}$$

- A unique stable trajectory exists if Z_{22} is square **and** non-singular.

Blanchard and Kahn's condition

A unique stable trajectory exists if there are as many roots larger than one in modulus as there are forward–looking variables in the model **and** the rank condition is satisfied.

First order approximation

Reduced form solution

- Finally, we have:

$$\hat{y}_t = g_y \hat{y}_{t-1} + g_u \epsilon_t$$
$$\Leftrightarrow y_t = (I_n - g_y) y^\star + g_y y_{t-1} + g_u \epsilon_t$$

  a VAR(1) model with a reduced rank covariance matrix (generally the model has less innovations than endogenous variables, $q < n$).

- The unconditional expectation of y_t is the deterministic steady state, $\mathbb{E}[y_t] = y^\star$. This is a manifestation of the certainty equivalence property.

- The unconditional covariance matrix, $\Sigma_y = \mathbb{V}[y_t]$, must solve:

$$\Sigma_y = g_y \Sigma_y g_y' + g_u \Sigma_\epsilon g_u'$$

  Specialized algorithms exist to solve efficiently this kind of equations... Otherwise the vec operator and kronecker product can be used:

$$\mathrm{vec}\Sigma_y = (I_{n^2} - g_y \otimes g_y)^{-1} \mathrm{vec} g_u \Sigma_\epsilon g_u'$$

First order approximation

Reduced form solution

- Inverting the reduced form, we obtain the MA(∞) representation:

$$\Leftrightarrow y_t = y^\star + \sum_{i=0}^{\infty} g_y^i g_u \epsilon_{t-i}$$

  a VAR(1) model with a reduced rank covariance matrix (generally the model has less innovations than endogenous variables, $q < n$).

- Let e_j be the j-th column of I_n

- The sequence $\{g_y^i g_u e_j\}_{i=0}^{\infty}$ is the IRF associated to a unitary shock on the j-th innovation.

- If the innovations are not orthogonal (which is a bad practice) a Cholesky decomposition can be used.

Higher order approximation

Introduction

- If the reduced form is (log)linear, the (approximated) behavior of the agents does not depend on future uncertainty.
- In such an environment we cannot reproduce the precautionary saving behavior, even if this behavior exists in the original nonlinear model.
- In the coming section, we show how to overcome this limit by considering higher order approximations.
- We only present the second order approximation. We will show that this is enough to disentangle the unconditional expectation and the deterministic steady state (break the certainty equivalence property inherent to the first order approximation).
- Higher order approximations (> 2) do not introduce additional algebraic complexities.

Second order approximation

Second order Taylor approximation

- With a second order Taylor expansion of F around $\bar{y}$:

$$
\begin{aligned}
F^{(2)}(y_-, u, u_+, \sigma) &= F^{(1)}(y_-, u, u_+, \sigma) \\
&+ \frac{1}{2}\left(F_{y_-y_-}(\hat{y} \otimes \hat{y}) + F_{uu}(u \otimes u) + F_{u_+u_+}(u_+ \otimes u_+) + F_{\sigma\sigma}\sigma^2\right) \\
&+ F_{y_-u}(\hat{y} \otimes u) + F_{y_-u_+}(\hat{y} \otimes u_+) + F_{y_-\sigma}\hat{y}\sigma \\
&+ F_{uu_+}(u \otimes u_+) + F_{u\sigma}u\sigma + F_{u_+\sigma}u_+\sigma
\end{aligned}
$$

- Taking the time t conditional expectation, we get:

$$
\begin{aligned}
0 \simeq \mathbb{E}_t &\left[F^{(1)}(y_-, u, u_+, \sigma)\right] \\
&+ \frac{1}{2}\left(F_{y_-y_-}(\hat{y} \otimes \hat{y}) + F_{uu}(u \otimes u) + F_{u_+u_+}(\sigma^2\vec{\Sigma}_\epsilon) + F_{\sigma\sigma}\sigma^2\right) \\
&+ F_{y_-u}(\hat{y} \otimes u) + F_{y_-\sigma}\hat{y}\sigma + F_{u\sigma}u\sigma
\end{aligned}
$$

$\Rightarrow$ We have six more unknowns: g_{yy}, g_{yu}, g_{uu}, $g_{y\sigma}$, $g_{u\sigma}$ and $g_{\sigma\sigma}$ (hidden in the second order derivatives of F).

Second order approximation

Second order derivatives

- The second order derivatives of a vector of multivariate functions is a three dimensional object. We use the following notation

$$\frac{\partial^2 F}{\partial x \partial x} = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1 \partial x_1} & \frac{\partial^2 F_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_1}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_1}{\partial x_n \partial x_n} \\ \frac{\partial^2 F_2}{\partial x_1 \partial x_1} & \frac{\partial^2 F_2}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_2}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_2}{\partial x_n \partial x_n} \\ \vdots & \vdots \ddots & \vdots & \ddots & \vdots & \\ \frac{\partial^2 F_m}{\partial x_1 \partial x_1} & \frac{\partial^2 F_m}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_m}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_m}{\partial x_n \partial x_n} \end{bmatrix}$$

- Let

$$\begin{aligned} y &= g(s) \\ f(y) &= f(g(s)) \end{aligned}$$

then the second order chain derivate rule is

$$\frac{\partial^2 f}{\partial s \partial s} = \frac{\partial f}{\partial y} \frac{\partial^2 g}{\partial s \partial s} + \frac{\partial^2 f}{\partial y \partial y} \left(\frac{\partial g}{\partial s} \otimes \frac{\partial g}{\partial s} \right)$$

Second order approximation

Recovering g_{yy}

- Assuming we have already solved for g_y, we must have:

$$\begin{aligned} F_{y_- y_-} &= f_{y_+}\left(g_{yy}(g_y \otimes g_y) + g_y g_{yy}\right) + f_y g_{yy} + \mathcal{B} \\ &= 0 \end{aligned}$$

  where $\mathcal{B}$ is a term that doesn't contain second order derivatives of function g.

- The equation can be rearranged:

$$\left(f_{y_+} g_y + f_y\right) g_{yy} + f_{y_+} g_{yy}(g_y \otimes g_y) = -\mathcal{B}$$

- This is a Sylvester type of equation and must be solved with an appropriate algorithm.

Second order approximation

Recovering g_{yu}

- We must have:

$$\begin{aligned} F_{y\_u} &= f_{y_+}\left(g_{yy}(g_y \otimes g_u) + g_y g_{yu}\right) + f_y g_{yu} + \mathcal{B} \\ &= 0 \end{aligned}$$

  where $\mathcal{B}$ is a term that doesn't contain second order derivatives of function g.

- This is a standard linear problem:

$$g_{yu} = -\left(f_{y_+} g_y + f_y\right)^{-1}\left(\mathcal{B} + f_{y_+} g_{yy}(g_y \otimes g_u)\right)$$

Second order approximation

Recovering g_{uu}

- We must have:

$$\begin{aligned} F_{uu} &= f_{y_+}\left(g_{yy}(g_u \otimes g_u) + g_y g_{uu}\right) + f_y g_{uu} + \mathcal{B} \\ &= 0 \end{aligned}$$

  where $\mathcal{B}$ is a term that doesn't contain second order derivatives of function g.

- This is a standard linear problem:

$$g_{uu} = -\left(f_{y_+} g_y + f_y\right)^{-1}\left(\mathcal{B} + f_{y_+} g_{yy}(g_u \otimes g_u)\right)$$

Second order approximation

Recovering $g_{y\sigma}$ and $g_{u\sigma}$

- We must have:

$$\begin{aligned} F_{y_-\sigma} &= f_{y_+} g_y g_{y\sigma} + f_y g_{y\sigma} \\ &= 0 \\ F_{u\sigma} &= f_{y_+} g_y g_{u\sigma} + f_y g_{u\sigma} \\ &= 0 \end{aligned}$$

  because we already established that $g_\sigma = 0$.

- Consequently

$$g_{y\sigma} = g_{u\sigma} = 0$$

- The size of the structural innovations does not affect the marginal effect of y_{t-1} and u_t on y_t.

- The last property would not resist if we consider a higher order approximation (> 2).

Second order approximation

Recovering $g_{\sigma\sigma}$

- We must have:

$$\begin{aligned} F_{\sigma\sigma} + F_{u_+u_+}\Sigma_\epsilon &= f_{y_+}\left(g_{\sigma\sigma} + g_y g_{\sigma\sigma}\right) + f_y g_{\sigma\sigma} \\ &\quad + \left(f_{y_+y_+}\left(g_u \otimes g_u\right) + f_{y_+} g_{uu}\right)\vec{\Sigma}_\epsilon \\ &= 0 \end{aligned}$$

  taking into account $g_\sigma = 0$.

- This is a standard linear problem:

$$g_{\sigma\sigma} = -\left(f_{y_+}\left(I + g_y\right) + f_y\right)^{-1}\left(f_{y_+y_+}\left(g_u \otimes g_u\right) + f_{y_+} g_{uu}\right)\vec{\Sigma}_\epsilon$$

- We have lost the certainty equivalence property!

Second order approximation

Reduced form solution

- The reduced form solution is augmented with quadratic terms:

$$y_t = y^\star + \frac{1}{2} g_{\sigma\sigma}\sigma^2 + g_y \hat{y} + g_u u + \frac{1}{2}\left(g_{yy}(\hat{y} \otimes \hat{y}) + g_{uu}(u \otimes u)\right) + g_{yu}(\hat{y} \otimes u)$$

  Where we fixed $\sigma = 1$.

- The unconditional variance consistent with a second order approximation is unchanged w.r.t what we obtained previously:

$$\Sigma_y = g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u'$$

  ie we omit the quadratic terms (which would involve third and fourth order terms in $\mathbb{E}[y_t y_t']$).

- The unconditional expectation is given by

$$\mathbb{E}\left[y_t\right] = y^\star + \left(I - g_y\right)^{-1}\left(\frac{1}{2}\left(g_{\sigma\sigma} + g_{yy}\vec{\Sigma}_y + g_{uu}\vec{\Sigma}_\epsilon\right)\right)$$

Second order approximation

Reduced form solution (explosive paths)

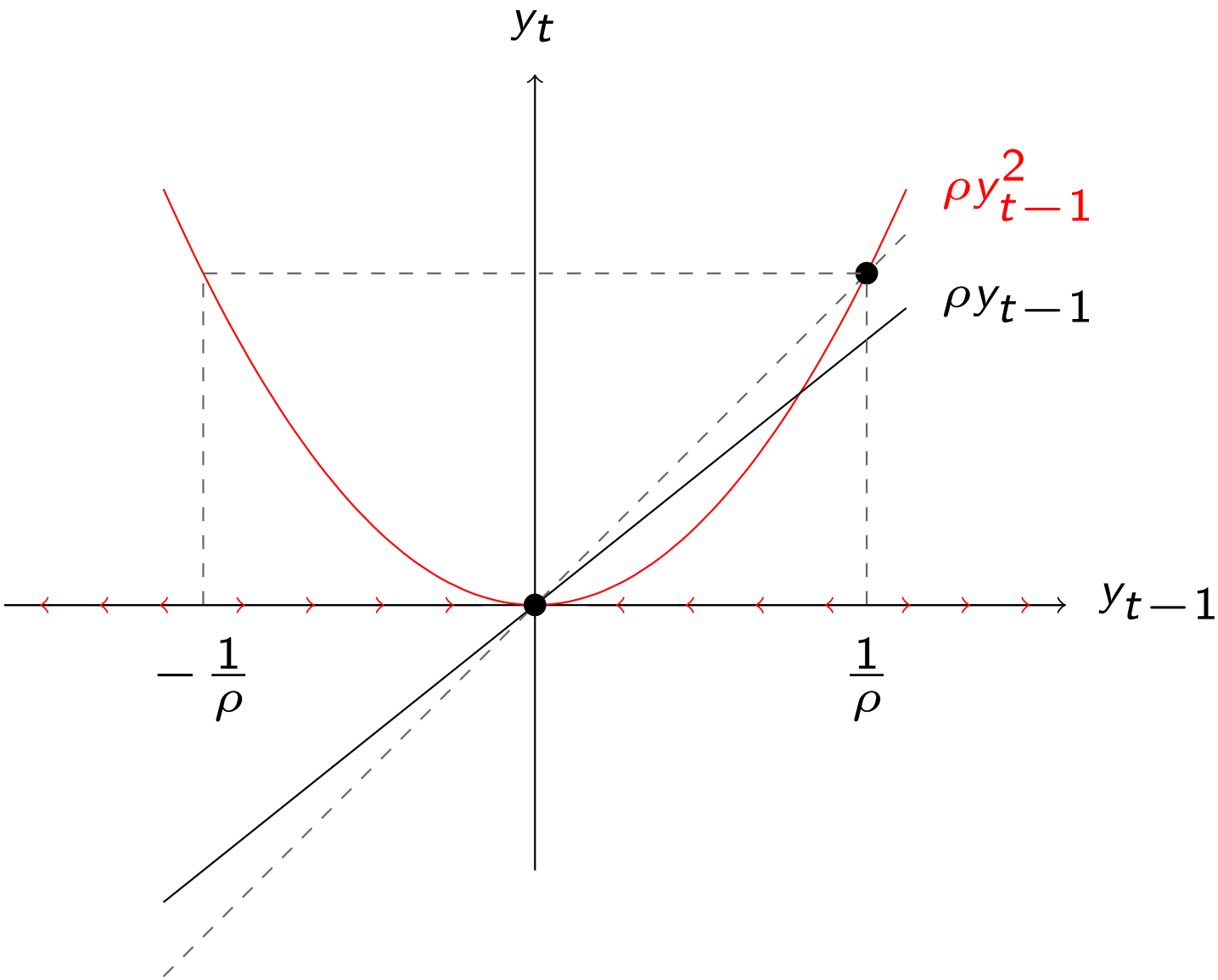
- Simulation of the endogenous variables (IRFs or time series) can result in explosive paths, even if the non approximated model is stable.
- This instability is caused by the quadratic terms in the second order reduced form.
- To get an intuition, compare a linear AR(1) and a quadratic AR(1):

$$y_t = \rho y_{t-1} + \epsilon_t$$
$$y_t = \rho y_{t-1}^2 + \epsilon_t$$

- The linear AR(1) has a unique deterministic steady state, $y^\star = 0$, globally stable provided that $|\rho| < 1$.
- The quadratic AR(1) share the same deterministic steady state, plus a "spurious" steady state $\bar{y} = {}^1/_\rho$.
- The first steady state is only locally stable while, the second one is unstable. Note that the local stability of $y^\star$ does not depend on the value of ρ.

Second order approximation

Quadratic vs. Linear AR(1) models



Second order approximation

Reduced form solution (explosive paths)

- In the quadratic AR(1), if y_t goes outside the interval $\left(-\frac{1}{\rho}, \frac{1}{\rho}\right)$, the generated times series will eventually diverge towards $+\infty$.

- One can easily show that if $y_0 = y^\star = 0$, then the IRFs, for the linear and quadratic AR(1) models, associated to an innovation ϵ_1 are respectively:

$$y_t = \rho^t \epsilon_1 \qquad \text{and} \qquad y_t = \frac{1}{\rho}\left(\rho\epsilon_1\right)^{2^{t-1}}$$

- Clearly, in the quadratic case, the IRF converges to $y^\star$ iff $|\epsilon_1| < 1/\rho$.

- More generally, the stability properties of time series generated by the quadratic AR(1) model depends on the entire history of innovations (path dependency).

Second order approximation

Reduced form solution (explosive paths and risky steady state)

- Obviously the second order reduced form of a DSGE model is not as simple as the quadratic AR(1) model.
- Next figure plots the transition equations associated to the first order (blue) and second order (red) approximations a DSGE model around the deterministic steady state $y^\star$.
- The transition equation associated to the second order approximation of the model has two fixed points: $\bar{y}$ (unstable) and $\tilde{y}$ (stable, because the slope of the transition equation is smaller than one at $\tilde{y}$). Both fixed points are different from the deterministic steady state of the original model.
- The magnitude of the jump of the transition equation at $y^\star$ is determined by $g_{\sigma\sigma}$ which characterizes the effect of future uncertainty.
- $\tilde{y}$ is called the risky steady state. The economy does not move away from $\tilde{y}$ if we take into account the possibility of future uncertainty.

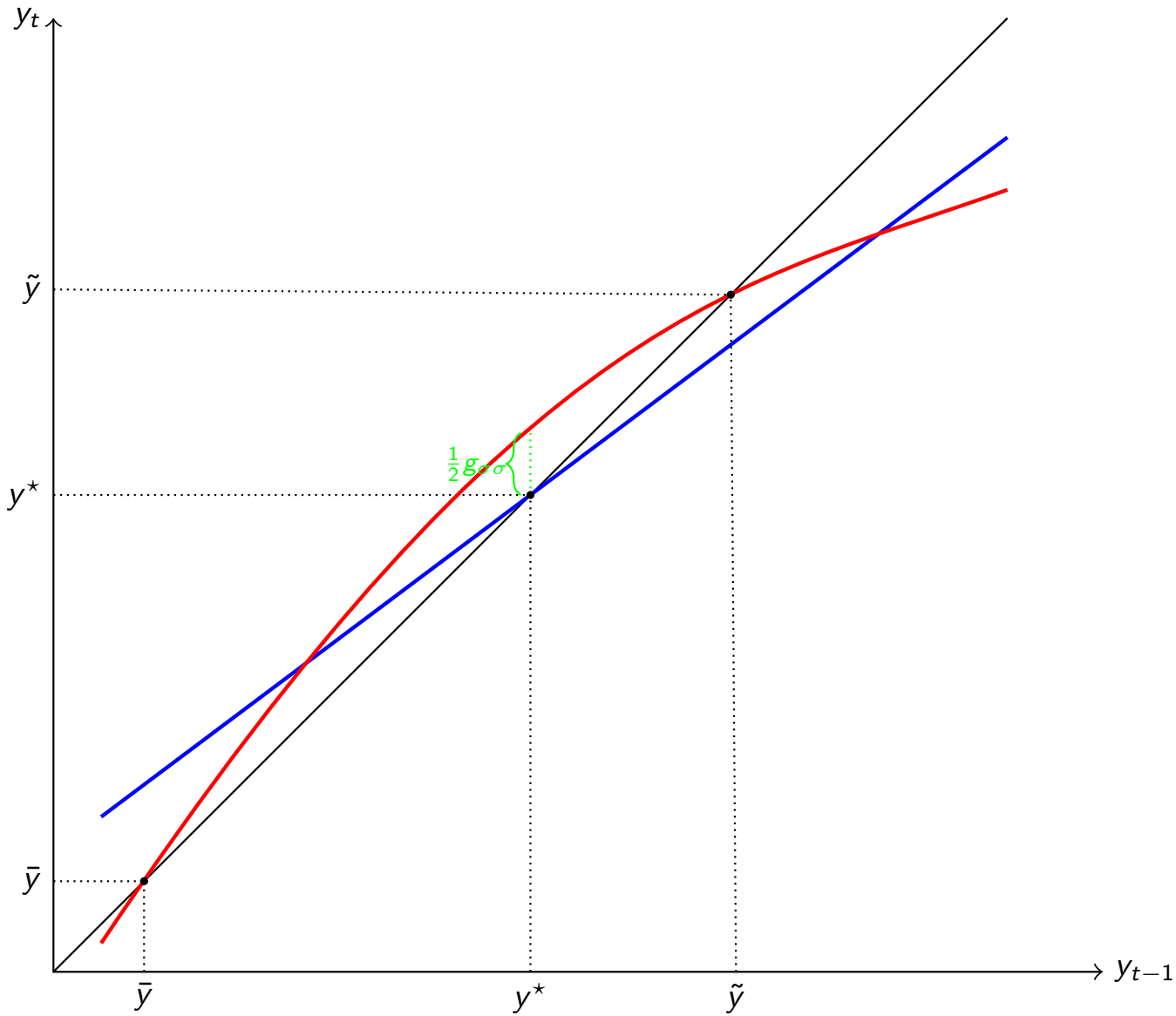
Second order approximation

Reduced form solution (explosive paths and risky steady state)

- The economy would not stay in $y^\star$, the deterministic steady state, if we take into account future uncertainty.
- Suppose that the plotted variable is the physical capital stock in an RBC model.
- The household decides to increase its saving as an insurance against future shocks $\Rightarrow$ The long run level of the physical capital stock is higher in an economy with uncertainty ($\tilde{y}$) than in a deterministic economy ($y^\star$).
- Because of this precautionary behavior, which is, at least partially, preserved by a second order approximation, the deterministic steady state, $y^\star$, cannot be a fixed point.
- The risky steady state is only locally stable. If y goes below $\bar{y}$, y will eventually diverge towards $-\infty$.

Second order approximation

Reduced form solution (explosive paths and risky steady state)



Second order approximation

Reduced form solution (pruning)

- Different strategies have been proposed to force the stability of the simulations. The more popular one was proposed by Kim, Kim, Schaumburg, and Sims: the pruning.
- Basically, the idea is to modify the recurrence by removing all the terms of order greater than two.
- This is done by replacing the second order reduced form by:

$$y_t = y^\star + \frac{1}{2} g_{\sigma\sigma}\sigma^2 + g_y \hat{y}_{t-1} + g_u u_t + \frac{1}{2}\left(g_{yy}(\hat{y}^0_{t-1} \otimes \hat{y}^0_{t-1}) + g_{uu}(u_t \otimes u_t)\right) + g_{yu}(\hat{y}^0_{t-1} \otimes u_t)$$

with

$$\hat{y}^0_t = g_y \hat{y}^0_{t-1} + g_u \epsilon_t$$

- Provided that $\hat{y}^0_t$ is stationary, pruned simulations $\{y_t\}$ will not explode (because we do not cumulate y_t or $\hat{y}^0_t$ through the second order terms).
- Note that the pruned model increases the number of states.

Perturbation methods with Dynare

- Dynare implements perturbation approximations of order 1, 2 and 3 (2 is the default).
- If higher order approximations are needed, use Dynare++
- The simulations are triggered by the `stoch_simul` command. See the manual for an exhaustive description of the options.
- The covariance matrix of the innovations must be specified before the call to `stoch_simul` using the `shocks` block.

Content of rbc1.mod

```
shocks;
    var LoggedProductivityInnovation = .01^2;
end;

stoch_simul(order=1,periods=1000);

figure('name','Policy rule');
plot(Capital,Consumption,'ok');
```

- If `periods>0` Dynare computes the simulated moments. If `periods=0`, which is the default value, Dynare reports the theoretical moments.

Perturbation methods with Dynare

- Dynare First reports a summary about the status of the variables in the model (number of predetermined variables, number of choice variables, ...) and prints the covariance matrix used for the simulations or computation of theoretical moments.
- Second Dynare prints the policy and transition equations obtained by solving the model.
- Third Dynare reports various descriptive statistics about the endogenous variables (covariance matrix, autocorrelation, ...).
- Dynare also computes Impulse Response Functions for each innovations.
- More output are available depending on the options passed to the `stoch_simul` command (see the manual).
- All the outputs can be accessed programmatically in the global Matlab structure `oo_` (see the manual again).