Local & Global Approximations of DSGE models

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Let y be a $n \times 1$ vector of endogenous variables, u is a $q \times 1$ vector of exogenous stochastic shocks. We consider the following type of model:

$$\mathbb{E}_t \left[f(y_{t+1}, y_t, y_{t-1}, u_t) \right] = 0$$

with:

$$\begin{aligned} u_t &= \sigma \epsilon_t \\ \mathbb{E}[\epsilon_t] &= 0 \\ \mathbb{E}[\epsilon_t \epsilon_t'] &= \Sigma_\epsilon \end{aligned}$$

where σ is a scale parameter, ϵ is a vector of auxiliary random variables.

Assumption $f: \quad \mathbb{R}^{3n+q} \rightarrow \mathbb{R}^n$ is a differentiable function in $\mathcal{C}^k$.

$$y_t = g(y_{t-1}, u_t, \sigma)$$

The unknown function g collects the policy rules and transition equations.

Then,

$$\begin{aligned} y_{t+1} &= g(y_t, u_{t+1}, \sigma) \\ &= g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma) \end{aligned}$$

and we define:

$$F_g(y_{t-1}, u_t, u_{t+1}, \sigma) = f(g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t)$$

Our problem may be then written as:

$$\mathbb{E}_t \left[F_g(y_{t-1}, u_t, u_{t+1}, \sigma)\right] = 0$$

- A deterministic steady state, $\bar{y}$, for the model satisfies

$$f(\bar{y}, \bar{y}, \bar{y}, 0) = 0$$

- A model can have several steady states, but only one of them will be used for approximation.
- Furthermore, the solution function satisfies:

$$\bar{y} = g(\bar{y}, 0, 0)$$

Let $\hat{y} = y_{t-1} - \bar{y}$, $u = u_t$, $u_+ = u_{t+1}$, $f_{y_+} = \frac{\partial f}{\partial y_{t+1}}$, $f_y = \frac{\partial f}{\partial y_t}$, $f_{y_-} = \frac{\partial f}{\partial y_{t-1}}$, $f_u = \frac{\partial f}{\partial u_t}$, $g_y = \frac{\partial g}{\partial y_{t-1}}$, $g_u = \frac{\partial g}{\partial u_t}$, $g_\sigma = \frac{\partial g}{\partial \sigma}$. Where all the derivates are evaluated at the deterministic steady state.

With a first order Taylor expansion of F around $\bar{y}$:

$$
\begin{aligned}
0 \simeq\ & F_g{}^{(1)}(y_-, u, u_+, \sigma) = \\
& f_{y_+}\left(g_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + g_u u_+ + g_\sigma\sigma\right) \\
& + f_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + f_{y_-}\hat{y} + f_u u
\end{aligned}
$$

What has changed? We now have three unknown "parameters" (g_y, g_u and g_σ) instead of an infinite number of parameters (function g).

Taking the expectation conditional on the information at time t, we have:

$$\begin{aligned} 0 \simeq & f_{y_+}\left(g_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + g_u\mathbb{E}_t[u_+] + g_\sigma\sigma\right) \\ & + f_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + f_{y_-}\hat{y} + f_u u \end{aligned}$$

or equivalently:

$$\begin{aligned} 0 \simeq & \left(f_{y_+}g_y g_y + f_y g_y + f_{y_-}\right)\hat{y} + \left(f_{y_+}g_y g_u + f_y g_u + f_u\right)u \\ & + \left(f_{y_+}g_y g_\sigma + f_{y_+}g_\sigma + f_y g_\sigma\right)\sigma \end{aligned}$$

This "equality" must hold for any value of $(\hat{y}, u, \sigma)$, so that the terms between parenthesis must be zero. We have three (multivariate) equations and three (multivariate) unknowns.

Let us assume that g_y is known. We must have:

$$f_{y_+} g_y g_\sigma + f_{y_+} g_\sigma + f_y g_\sigma = 0$$

Solving for g_σ, we obtain

$$g_\sigma = 0$$

This is a manifestation of the certainty equivalence property of first order approximation: the policy rule does not depend on the size of the structural shocks.

Let us assume again that g_y is known. We must have:

$$f_{y_+} g_y g_u + f_y g_u + f_u = 0$$

Solving for g_u, we obtain

$$g_u = -\left(f_{y_+} g_y + f_y\right)^{-1} f_u$$

g_u gives the marginal effect of the structural innovations on the endogenous (jumping and states) variables.

We must have:

$$\left(f_{y_+} g_y g_y + f_y g_y + f_{y_-}\right) \hat{y} = 0$$

Structural state space representation:

$$\begin{bmatrix} 0 & f_{y+} \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} g_y \hat{y} = \begin{bmatrix} -f_{y_-} & -f_y \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} \hat{y}$$

or

$$\begin{bmatrix} 0 & f_{y+} \\ I & 0 \end{bmatrix} \begin{bmatrix} y_t - \bar{y} \\ y_{t+1} - \bar{y} \end{bmatrix} = \begin{bmatrix} -f_{y_-} & -f_y \\ 0 & I \end{bmatrix} \begin{bmatrix} y_{t-1} - \bar{y} \\ y_t - \bar{y} \end{bmatrix}$$

Because g_y is the marginal effect of y_{t-1} on y_t.

$$Dx_{t+1} = Ex_t$$

with

$$x_{t+1} = \begin{bmatrix} y_t - \bar{y} \\ y_{t+1} - \bar{y} \end{bmatrix} \qquad x_t = \begin{bmatrix} y_{t-1} - \bar{y} \\ y_t - \bar{y} \end{bmatrix}$$

- There is an infinity of solutions but we want a unique stable one.

- Matrix D may be singular.

Taking the real generalized Schur decomposition of the pencil $< E, D >$:

$$\begin{aligned} D &= QTZ \\ E &= QSZ \end{aligned}$$

with T, upper triangular, S quasi-upper triangular, $Q'Q = I$ and $Z'Z = I$.

DEFINITION: **Generalized eigenvalues**

λ_i solves

$$\lambda_i D v_i = E v_i$$

For diagonal blocks on S of dimension 1 x 1:

- $T_{ii} \neq 0$: $\lambda_i = \frac{S_{ii}}{T_{ii}}$
- $T_{ii} = 0$, $S_{ii} > 0$: $\lambda = +\infty$
- $T_{ii} = 0$, $S_{ii} < 0$: $\lambda = -\infty$
- $T_{ii} = 0$, $S_{ii} = 0$: $\lambda \in \mathcal{C}$

Applying the Schur decomposition and multiplying by Q' we obtain:

$$
\begin{aligned}
D\begin{bmatrix} I \\ g_y \end{bmatrix} g_y\hat{y} &= E\begin{bmatrix} I \\ g_y \end{bmatrix}\hat{y} \\
\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I \\ g_y \end{bmatrix} g_y\hat{y} \\
&= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I \\ g_y \end{bmatrix}\hat{y}
\end{aligned}
$$

Where S and T are arranged in such a way that the stable eigenvalues comes first.

g_y is recovered by selecting the stable path. To exclude explosive trajectories, one must impose

$$Z_{21} + Z_{22} g_y = 0$$

or equivalently:

$$g_y = -Z_{22}^{-1} Z_{21}$$

A unique stable trajectory exists if Z_{22} is square **and** non-singular. With Blanchard and Kahn's words: there are as many roots larger than one in modulus as there are forward–looking variables in the model **and** the rank condition is satisfied.

With a second order Taylor expansion of F around $\bar{y}$:

$$F^{(2)}(y_-, u, u_+, \sigma) = F^{(1)}(y_-, u, u_+, \sigma)$$
$$+ \frac{1}{2}\left(F_{y_-y_-}(\hat{y} \otimes \hat{y}) + F_{uu}(u \otimes u) + F_{u_+u_+}(u_+ \otimes u_+) + F_{\sigma\sigma}\sigma^2\right)$$
$$+ F_{y_-u}(\hat{y} \otimes u) + F_{y_-u_+}(\hat{y} \otimes u_+) + F_{y_-\sigma}\hat{y}\sigma$$
$$+ F_{uu_+}(u \otimes u_+) + F_{u\sigma}u\sigma + F_{u_+\sigma}u_+\sigma$$

and taking the time t conditional expectation, we get:

$$0 \simeq \mathbb{E}_t\left[F^{(1)}(y_-, u, u_+, \sigma)\right]$$
$$+ \frac{1}{2}\left(F_{y_-y_-}(\hat{y} \otimes \hat{y}) + F_{uu}(u \otimes u) + F_{u_+u_+}(\sigma^2\vec{\Sigma}_\epsilon) + F_{\sigma\sigma}\sigma^2\right)$$
$$+ F_{y_-u}(\hat{y} \otimes u) + F_{y_-\sigma}\hat{y}\sigma + F_{u\sigma}u\sigma$$

We have six more unknowns: g_{yy}, g_{yu}, g_{uu}, $g_{y\sigma}$, $g_{u\sigma}$ and $g_{\sigma\sigma}$.

The second order derivatives of a vector of multivariate functions is a three dimensional object. We use the following notation

$$\frac{\partial^2 F}{\partial x \partial x} = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1 \partial x_1} & \frac{\partial^2 F_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_1}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_1}{\partial x_n \partial x_n} \\ \frac{\partial^2 F_2}{\partial x_1 \partial x_1} & \frac{\partial^2 F_2}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_2}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_2}{\partial x_n \partial x_n} \\ \vdots & \vdots \ddots & \vdots & \ddots & \vdots & \\ \frac{\partial^2 F_m}{\partial x_1 \partial x_1} & \frac{\partial^2 F_m}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F_m}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F_m}{\partial x_n \partial x_n} \end{bmatrix}$$

Let

$$\begin{aligned} y &= g(s) \\ f(y) &= f(g(s)) \end{aligned}$$

then,

$$\frac{\partial^2 f}{\partial s \partial s} = \frac{\partial f}{\partial y} \frac{\partial^2 g}{\partial s \partial s} + \frac{\partial^2 f}{\partial y \partial y} \left(\frac{\partial g}{\partial s} \otimes \frac{\partial g}{\partial s} \right)$$

Assuming we have already solved for g_y, we must have:

$$\begin{aligned} F_{y_-y_-} &= f_{y_+}\left(g_{yy}(g_y \otimes g_y) + g_y g_{yy}\right) + f_y g_{yy} + \mathcal{B} \\ &= 0 \end{aligned}$$

where $\mathcal{B}$ is a term that doesn't contain second order derivatives of function g.

The equation can be rearranged:

$$\left(f_{y_+} g_y + f_y\right) g_{yy} + f_{y_+} g_{yy}(g_y \otimes g_y) = -\mathcal{B}$$

This is a Sylvester type of equation and must be solved with an appropriate algorithm.

We must have:

$$\begin{aligned} F_{y_-u} &= f_{y_+}\left(g_{yy}(g_y \otimes g_u) + g_y g_{yu}\right) + f_y g_{yu} + \mathcal{B} \\ &= 0 \end{aligned}$$

where $\mathcal{B}$ is a term that doesn't contain second order derivatives of function g.

This is a standard linear problem:

$$g_{yu} = -\left(f_{y_+} g_y + f_y\right)^{-1}\left(\mathcal{B} + f_{y_+} g_{yy}(g_y \otimes g_u)\right)$$

We must have:

$$\begin{aligned} F_{uu} &= f_{y_+}\left(g_{yy}(g_u \otimes g_u) + g_y g_{uu}\right) + f_y g_{uu} + \mathcal{B} \\ &= 0 \end{aligned}$$

where $\mathcal{B}$ is a term that doesn't contain second order derivatives of function g.

This is a standard linear problem:

$$g_{uu} = -\left(f_{y_+} g_y + f_y\right)^{-1}\left(\mathcal{B} + f_{y_+} g_{yy}(g_u \otimes g_u)\right)$$

We must have:

$$\begin{aligned} F_{y_-\sigma} &= f_{y_+} g_y g_{y\sigma} + f_y g_{y\sigma} \\ &= 0 \\ F_{u\sigma} &= f_{y_+} g_y g_{u\sigma} + f_y g_{u\sigma} \\ &= 0 \end{aligned}$$

as $g_\sigma = 0$. Then:

$$g_{y\sigma} = g_{u\sigma} = 0$$

The size of the structural innovations do not affect the marginal effect of y_{t-1} and u_t on y_t.

We must have:

$$\begin{aligned} F_{\sigma\sigma} + F_{u_+u_+}\Sigma_\epsilon &= f_{y_+}\left(g_{\sigma\sigma} + g_y g_{\sigma\sigma}\right) + f_y g_{\sigma\sigma} \\ &\quad + \left(f_{y_+y_+}(g_u \otimes g_u) + f_{y_+} g_{uu}\right)\vec{\Sigma}_\epsilon \\ &= 0 \end{aligned}$$

taking into account $g_\sigma = 0$.

This is a standard linear problem:

$$g_{\sigma\sigma} = -\left(f_{y_+}(I + g_y) + f_y\right)^{-1}\left(f_{y_+y_+}(g_u \otimes g_u) + f_{y_+} g_{uu}\right)\vec{\Sigma}_\epsilon$$

We have lost the certainty equivalence property!

$$y_t = \bar{y} + 0.5 g_{\sigma\sigma}\sigma^2 + g_y \hat{y} + g_u u + 0.5 \left(g_{yy}(\hat{y} \otimes \hat{y}) + g_{uu}(u \otimes u) \right) + g_{yu}(\hat{y} \otimes u)$$

We can fix $\sigma = 1$.

Second order accurate moments:

$$\begin{aligned} \Sigma_y &= g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u' \\ \mathbb{E}\left[y_t\right] &= \bar{y} + (I - g_y)^{-1} \left(0.5 \left(g_{\sigma\sigma} + g_{yy} \vec{\Sigma}_y + g_{uu} \vec{\Sigma}_\epsilon \right) \right) \end{aligned}$$

- For large shocks second order approximation simulation may explode
  - pruning algorithm (Sims)
  - truncate normal distribution (Judd)
- The model has to be defined by $f \in \mathcal{C}^k$.
- The approximated solution is local so we cannot analyse transitions from one steady state to another.

A global approximation of the unknown function g is needed...

But to keep things tractable we also need (somehow) to "project" this infinite dimensional problem in a finite dimensional space.

- Suppose we want to solve the following dynamic problem:

$$\dot{y}(t) = y(t)$$

  for $t \in [0, T]$ given the initial condition $y(0) = 1$ (backward looking dynamic) where $y \in \mathcal{C}^1$.

- This problem is trivial, the solution is $y(t) = e^t$, but let us assume that we don't know how to solve a differential equation.

- Define the operator $\mathcal{L}$ by

$$\mathcal{L}f \equiv \dot{f} - f$$

  a mapping from $\mathcal{C}^1$ to $\mathcal{C}^0$.

- The dynamic problem can now be formulated as a "fixed point" problem in the space $\mathcal{C}^1$ of functions:

$$\mathcal{L}y = 0$$

- The idea is to approximate the unknown y by a weighted sum of monomial terms on the interval $[0, T]$:

$$\hat{y}(t; \mathbf{a}) \equiv a_0 + \sum_{i=1}^{n} a_i t^i$$

  where obviously $a_0 = 1$ (to fit the initial condition), so we redefine $\mathbf{a} \equiv (a_1, \ldots, a_n)'$. Our infinite dimensional problem is reduced to a finite dimensional problem with n unknowns. We just need to find $\mathbf{a} \in \mathbb{R}^n$ which provides an acceptable approximation of y.

Theorem (Weierstrass) If $f \in \mathcal{C}[a,b]$, then for all $\epsilon > 0$, there exist a polynomial $p(x)$ such that:

$$\forall x \in [a,b], \quad |f(x) - p(x)| \leq \epsilon$$

If $f \in \mathcal{C}^k[a,b]$, then there exists a sequence of order n polynomials, p_n, such that:

$$\lim_{n \to \infty} \max_{x \in [a,b]} |f^{(l)}(x) - p_n^{(l)}(x)| = 0$$

for $l \leq k$.

- Let:

$$R(t;\mathbf{a}) \equiv \mathcal{L}\hat{y}(t;\mathbf{a})$$
$$= a_1 - 1 + \sum_{i=2}^{n} a_i i t^{i-1} - \sum_{i=1}^{n} a_i t^i$$

be a residual function defined for $t \in [0, T]$.

- The idea is to choose **a** to make the residual as small as possible (given n).

- A first try would be:

$$\mathbf{a} = \arg\min_{\mathbf{a}} \int_0^T R(t,\mathbf{a})\mathrm{d}t$$

$\rightarrow$ NLLS, L^2 norm.

- Other approaches can be considered:
  - **Method of collocation:** $\mathbf{a} \in \mathbb{R}^n$ is such that the residual is exactly zero on n points $\{t_i\}_{i=1}^n$ in $[0, T]$:

$$R(t_i; \mathbf{a}) = 0 \text{ for } i = 1, \dots, n$$

  - **Method of moments:** the true solution is such that for any arbitrary function $p(t)$ we have $\int_0^T p(t)\mathcal{L}y(t)\mathrm{d}t = 0$. We can choose $\mathbf{a} \in \mathbb{R}^n$ so that for a sequence of arbitray functions $\{p_i(t)\}_{i=1}^n$ we have exactly:

$$\int_0^T p_i(t)\mathcal{L}\hat{y}(t; \mathbf{a})\mathrm{d}t \equiv \int_0^T p_i(t)R(t; \mathbf{a})\mathrm{d}t = 0$$

    Usually we consider $p_i(t) = t^i$.

- To solve a differential equation we need:
  - To express it as a zero of some operator.
  - To define a parameterized approximation function as a weighted sum of simple functions.
  - To identify the parameters of the approximation function by matching some conditions fullfilled by the true solution.
- Remaining issues:
  - What is the value of n?
  - What kind of “simple functions” should be choosen?
  - How should we solve for the parameters of the approximation function?

- Suppose that the unknown function is the solution of the following operator equation:

$$\mathcal{N}(f)$$

where $\mathcal{N} : B \to B$, B is a Banach space of functions $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$.

- **Trivial example.** In the previous example we have $D = [0, T]$, $f : D \to \mathbb{R}$ and

$$\mathcal{N} = \frac{\mathrm{d}}{\mathrm{d}t} - I$$

where I is the identity operator.

- **Economic example.** For a discrete time Ramsey growth model, D is the space of the state variable (capital stock k), the unknown function f is the policy rule (consumption as a function of k, $c(k)$), $\mathcal{N}$ is the Euler equation (so we have $n = m = 1$):

$$\mathcal{N}(c) \equiv u'(c(k)) - \beta u'\big(c(h(k) - c(k))\big)\Big(f'\big(h(k) - c(k)\big)\Big)$$

where $h(k) = f(k) + (1 - \delta)k$, u is the utility function and f is the production function.

1. Choose a basis $\Phi = \{\varphi_i\}_{i=1}^n$ and an inner product: $<\varphi_i, \varphi_j> = \int_D \varphi_i(x)\varphi_j(x)\omega(x)\mathrm{d}x$.
2. Choose a degree of approximation: n.
$$\hat{f} = \sum_{i=1}^{n} a_i \varphi_i(x)$$
3. For a guess $\mathbf{a}_j$ evaluate the approximation of f and the residual:
$$R(x; \mathbf{a}_j) = \left(\mathcal{N}(\hat{f})\right)(x)$$
4. Choose a sequence of n functions, $p_i : D \rightarrow \mathbb{R}^m$ and for each guess of $\mathbf{a}_j$ evaluate the n projections:
$$P_i = < R(., \mathbf{a}), p_i(.) >$$

- Each element, φ_i, of the basis should be simple to compute.
- Elements of the basis should be similar in size.
- Each element of the basis should bring a specific information.
- Ideally φ_i is orthogonal to φ_j with respect to the chosen inner product

$$< \varphi_i, \varphi_j >= 0$$

- Defined over [-1,1] by:

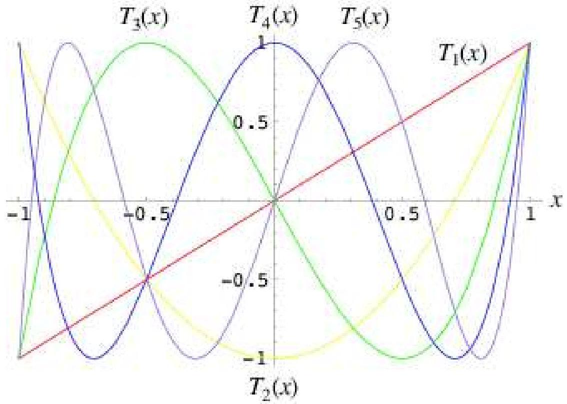
$$T_n(x) \equiv \cos(n \arccos x)$$

- Can be evaluated using the folowing recursion:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$



$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1
\end{aligned}
$$

- The Chebyshev polynomials are orthogonal with respect to the inner product defined by the weighting function $(1-x^2)^{-\frac{1}{2}}$:

$$\int_{-1}^{1} T_n(x)T_m(x)\,\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m = 0 \\ \pi/2 & \text{if } n = m \neq 0 \end{cases}$$

- Let z be a root of the order n Chebyshev polynomial, $T_n(z) = 0$. The n zeroes of T_n are given by:

$$z_{n,h} = \cos\left(\frac{(2h-1)\pi}{2n}\right) \text{ for } h = 1, \dots, n$$

- Suppose $f \in \mathcal{C}^k[a,b]$.
- Define:

$$c_j = \frac{2}{n} \sum_{k=1}^{n} f(z_{n,k}) T_j(z_{n,k})$$

and

$$\hat{f}_n(x) = -\frac{1}{2} c_0 + \sum_{k=1}^{n} c_k T_k(x)$$

- There exists some d_k such that for all n

$$||f - \hat{f}||_\infty \leq \left(\frac{2}{\pi} \log(n+1) + 2 \right) \frac{d_k}{n^k} ||f^{(k)}||_\infty$$

- If we have more than one state variable:
  - Tensor product basis.
  - Complete basis.
- Curse of dimensionality...

- We have to choose **a** which makes the residual small.
- NNLS (L^2 norm of the residuals) is a natural choice.
- More generaly, define the inner product as

$$< f, g >= \int_D f(x)g(x)w(x)\mathrm{d}x$$

where $w(x)$ is an arbitrary waighting function.

- If **a** solves NNLS then we have:

$$\mathbf{a} = \max_{\vec{a}} < R(., \vec{a}), R(., \vec{a}) >$$

with $w(x) = 1$ for all x.

- Aternatively we can fix n projections and and choose **a** such that the resulting residual in each of these n projections is zero.
- NNLS $\leftrightarrow$ GMM. So we choose **a** such that

$$\left\langle R(,\mathbf{a}), \frac{\partial R}{\partial a_i}(.,\mathbf{a}) \right\rangle = 0$$

for $i = 1, \ldots, n$.

- More generally we can replace the partial derivates of R by any collection of arbitrary function $\{p_i\}_{i=1}^n$ and choose **a** such that:

$$\langle R, p_i \rangle = 0$$

for $i = 1, \ldots, n$.

Choice of a projection condition (III)

- Galerkin
- Method of moments, collocation,...
- Orthogonal collocation.