Assessing long run risk in a DSGE model under ZLB with the stochastic extended path approach

Stéphane Adjemian*and Michel Juillard†

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Abstract

Assessing long run endogenous risk in a DSGE implies to take seriously both non-linearities and future stochastic shocks. Furthermore, non-binding constraints such as the zero lower bound for nominal interest rates (ZLB) make it difficult to use traditional perturbation methods even of higher order. We use instead a hybrid stochastic extended path approach. The extended path approach uses an auxiliary perfect foresight model to compute the effect of random shocks period by period. The stochastic extended path approach uses quadrature and several auxiliary perfect foresight models to compute numerically the conditional expectation in the nonlinear model for a few periods forward. The hybrid stochastic extended path approach uses a perturbation approach to take into account the long run effect of future random shocks that is not taken into account by the successice quadratures. We apply this approach to the model of Rudebusch and Swanson (2008). We find that the ZLB affects the risk premium when the ZLB is binding, but not such much outside of these episodes.

Introduction[1]

There is now an abundant literature on the macroeconomic determinants of asset prices and of term premium in particular. In that context term premia are compensation for consumption and inflation risks in the future.

*GAINS, Université du Maine and CEPREMAP,stephane.adjemian@univ-lemans.fr, http://www.dynare.org/stepan

†Banque de France and CEPREMAP, michel.juillard@mjui.fr.

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In a DSGE model, these risks unfold from the stochastic nature of the environment and the shocks affecting consumption and real returns of various class of assets. These risks affect agents' current behavior and asset prices because of the joint effect of nonlinearities and future uncertainty. They can't be addressed by (log-)linearizing the entire model because of the certainty equivalence property of linear models, nor by considering a perfect foresight, deterministic, version of that model.

If the model is small enough, it can be solved by iteration on the value function or a similar method, but even small DSGE models have too many state variables for this option to be practicable. Jermann (1998) log-linearizes the real part of the model and applies exact log normal formulas to determine asset prices and term premia. Rudebusch and Swanson (2008, 2012) use a third order perturbation approach.

However, local approximation doesn't permit to treat easily occasionally binding constraint and this makes difficult to discuss the effect of the zero lower bound for nominal interest rates on asset prices dynamics. In this paper, we use a *hybrid stochastic extended path method* to address this issue.

We simulate the model in Rudebusch and Swanson (2008) while imposing the ZLB. This is a standard DSGE model using Epstein-Zin preferences in order to disentangle risk aversion and inter temporal substitution. The model includes nominal rigidities in the form of Calvo pricing and is extended with a long-term bond in the form of a console calibrated to mimic a 10-year government bond. The term premium tracks the difference between the yield of this console and a risk free rate.

We find that, as expected, taking properly into account future uncertainty has important implication for the term premium. Because the ZLB puts a floor on the diminution of the yield, the presence of the ZLB diminishes the term premium in comparison with an hypothetical world where nominal interest rate could become negative.

In the first section, we present the main features of the Rudebusch and Swanson (2008) model. The different components of the hybrid stochastic extended path are detailed in section 2. Numerical results are discussed in section 3. Directions for future work conclude the paper.

1 Rudebusch and Swanson (2008) model with Epstein-Zin preferences

Rudebusch and Swanson (2008) model is a standard DSGE model with Epstein-Zin preferences, augmented with prices of nominal bonds and term premium.

1.1 Households

Households maximize expected utility provided by consumption c_t and, negatively, by labor effort ℓ_t, in such a way that their current welfare, W_t is given by:

$$W_t = u(c_t, \ell_t) + \beta \left(\mathbb{E}_t W_{t+1}^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \tag{1}$$

if $u(c_t, \ell_t) > 0$ everywhere, and by:

$$W_t = u(c_t, \ell_t) - \beta \left(\mathbb{E}_t (-W_{t+1})^{1-\alpha} \right)^{\frac{1}{1-\alpha}} \tag{2}$$

if $u(c_t, \ell_t) > 0$ everywhere (Epstein and Zin, 1989).

The Epstein-Zin utility specification breaks the equivalence between the inverse of the inter temporal elasticity of substitution and the coefficient of relative risk aversion that is unavoidable in the standard expected utility framework.

Period utility is defined as:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \chi_0 \frac{\ell_t^{1+\chi}}{1+\chi}.$$

If $\gamma > 1$, as it will be the case in the paper, u is negative everywhere and we will use equation (2) for the remaining of the paper.

The resource constraint faced by the households is:

$$p_t a_t + P_t c_t = w_t \ell_t + d_t + p_t a_{t-1}$$

where a_t is the stock of a state contingent asset at the end of period t with price p_t, the price of consumption is noted P_t and w_t is the nominal wage rate.

Households choose plans for a_t, c_t and ℓ_t, taking prices as given. The first order conditions of this optimization problem are:

$$\frac{(1-\chi)\ell_t^{\chi}}{(1-\gamma)c_t^{-\gamma}} = \frac{w_t}{P_t} \tag{3}$$

$$c_t^{-\gamma} = \beta \mathbb{E}_t \left(\mathbb{E}_t W_{t+1}^{1-\alpha} \right)^{\frac{\alpha}{1-\alpha}} W_{t+1}^{-\alpha} c_{t+1}^{-\gamma} (1 + r_{t+1}) \frac{P_t}{P_{t+1}} \tag{4}$$

In turn, the households stochastic discount factor at time t for a (stochastic) payoff at time $t+1$ is:

$$m_{t+1} = \left(\frac{W_{t+1}}{\left(\mathbb{E}_t W_{t+1}^{1-\alpha} \right)^{\frac{1}{1-\alpha}}} \right)^{\alpha} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \frac{P_t}{P_{t+1}} \quad . \tag{5}$$

1.2 Firms

There exist a continuum of monopolistically competitive firms producing intermediary goods indexed by $i \in [0, 1]$. These firms set prices and hires labor in a competitive labor market.

The production function for firm i is:

$$y_t(i) = A_t \bar{k}^{1-\eta} \ell_t(i)^\eta \tag{6}$$

where $\bar{k}$ is the fixed, firm specific capital stock and A_t is total factor productivity, common to all firms, and that follows:

$$\log A_t = \rho_A \log A_{t-1} + \varepsilon_t^A \quad , \tag{7}$$

where ε_t^A is an i.i.d. shock with mean zero and variance σ_A^2.

Firms set prices according to a Calvo lottery with probability $1-\xi$. The price chosen by the firms that have the opportunity of setting their price in period t is denoted p_t. There is no indexation and this price remains until the firm gets the opportunity to re–optimize.

When choosing a price, firm i maximizes the value to shareholders (the households) of the cash flow over the duration of that price:

$$\mathbb{E}_t \sum_{j=0}^{\infty} \xi^j m_{t,t+j} \left[p_t(i) y_{t+j}(i) - w_{t+j} \ell_{t+j}(i) \right] \quad , \tag{8}$$

where the stochastic discount process between periods t and $t+j$, $m_{t,t+j} = \Pi_{k=1}^{j} m_{t+k}$.

The final good sector is made of perfectly competitive firms that aggregate the continuum of intermediate products into a single final good using a CES technology:

$$Y_t = \left[\int_0^\infty y_t(i)^{\frac{1}{1+\theta}} di \right]^{1+\theta} \quad . \tag{9}$$

The demand curve for each intermediary firm is:

$$y_t(i) = \left(\frac{p_t(i)}{P_t} \right)^{-\frac{1+\theta}{\theta}} Y_t \quad , \tag{10}$$

and the aggregate price, P_t, is given by:

$$P_t = \left[\int_0^\infty p_t(i)^{-\frac{1}{\theta}} di \right]^{-\theta} \quad . \tag{11}$$

Solving the optimization problem for the firm gives the following optimality condition:

$$p_t(i) = \frac{(1+\theta) \mathbb{E}_t \sum_{J=0}^{\infty} \xi^j m_{t,t+j} mc_{t+j}(i) y_{t+j}(i)}{\mathbb{E}_t \sum_{J=0}^{\infty} \xi^j m_{t,t+j} y_{t+j}(i)} \quad , \tag{12}$$

where mc_t indicates the marginal cost for firm i at period t:

$$mc_t(i) = \frac{w_t \ell_t(i)}{\eta y_t(i)} \tag{13}$$

1.3 Aggregate supply and labor demand

We consider the following index of cross-sectional price dispersion:

$$\Delta_t^{\frac{1}{\eta}} = (1-\xi) \sum_{j=0}^{\infty} p_{t-j}(i)^{-\frac{1+\theta}{\theta\eta}} \quad . \tag{14}$$

The aggregate quantity of labor demanded by firms is:

$$L_t = \int_0^{\infty} \ell_t(i) di \quad . \tag{15}$$

In equilibrium, aggregate labor demanded by firms, L_t must equal ℓ_t, labor offered by the representative household.

Aggregate supply of final good, Y_y is such that:

$$Y_t = \Delta_t^{-1} A_t \overline{K}_t^{1-\eta} L_t^{\eta} \quad , \tag{16}$$

where $\overline{K} = \overline{k}$, the aggregate stock of capital.

1.4 Aggregate demand

Government consumption of final good follows an exogenous AR(1) process:

$$G_t = \rho_G G_{t-1} + \varepsilon_t^G \quad , \tag{17}$$

where ε_t^G is an i.i.d.shock with zero mean and variance σ_G^2. The government finances these expenditures by a lump-sum tax.

A fraction $\delta\overline{K}$ is set aside each period to maintain the capital stock. The aggregate resource constraint is then:

$$Y_r = C_t + \delta\overline{K} + G_t \quad , \tag{18}$$

where $C_t = c_t$ is the consumption of the representative household.

1.5 Monetary policy

The monetary authority follows a Taylor-type policy rule:

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left[\frac{1}{\beta} + \overline{\pi}_t + g_y \frac{Y_t - \overline{Y}}{\overline{Y}} - g_\pi(\overline{\pi}_t - \pi^\star)\right] + \varepsilon_t^i \quad . \tag{19}$$

In the above equation, $1/\beta$ is the steady-state real interest rate, $\overline{Y}$, the steady-state level of output, $\pi^\star$, the inflation target and ε_t^i and i.i.d shock to monetary policy with mean zero and variance σ_i^2. The variable $\overline{\pi}_t$ denotes a geometric moving average of inflation:

$$\overline{\pi}_t = \theta_p \overline{\pi}_{t-1} + (1-\theta_p)\pi_t \tag{20}$$

1.6 Long-term bonds and term premium

We consider a default-free nominal console which pays a geometrically declining coupon in every period to perpetuity. It's price per one dollar of coupon in period t, $\widetilde{p}^{(n)}$, satisfies:

$$\widetilde{p}_t^{(n)} = 1 + \delta_c \mathbb{E}_t m_{t+1} p_{t+1}^{(n)} \quad , \tag{21}$$

where δ_c is the rate of decay of the console's coupon. δ_c is calibrated such that the Macaulay duration of the console be equal to 10 years, so as to mimic the ten-year Treasury note. Working with a console is more convenient from a computational point of view than with a n-period bond, when n is larger than a few periods.

This default-free bond is still risky because its price covaries with the household's marginal utility of consumption. The term premium is then defined as the difference between the yield of a bond and the yield of a notional risk-free bond. The risk-neutral price of the console, $\widehat{p}_t^{(n)}$, is:

$$\widehat{p}_t^{(n)} = \mathbb{E}_t \sum_{j=0}^{\infty} e^{-i_{t,t+j}} \delta_c^j \tag{22}$$

with $i_{t,t+j} = \sum_{n=0}^{j} i_n$, and the term premium $\psi_t^{(n)}$ is defined as:

$$\psi_t^{(n)} = \log\left(\frac{\delta_c \widetilde{p}_t^{(n)}}{\widetilde{p}_t^{(n)} - 1}\right) - \log\left(\frac{\delta_c \widehat{p}_t^{(n)}}{\widehat{p}_t^{(n)} - 1}\right) \tag{23}$$

2 Numerical approximation

The only way to study the quantitative implication of this type of models is to resort to numerical simulations. Because of the occasionally binding nature of the zero lower bond for nominal interest rates, it is not possible to apply local approximation in the way done by Rudebusch and Swanson (2008). We propose to use instead an *hybrid stochastic extended path approach*.

Three elements are at the center of this approach: 1) an algorithm for deterministic models handles the nonlinearities with a high level of accuracy; 2) the conditional expectation at the heart of the stochastic model is approximated by a quadrature formula; 3) the risky steady state replaces the deterministic steady state as terminal value for the perfect foresight model as way to take into account level effects of future uncertainty. We review below the various building blocks of the approach.

The general form of the stochastic model that we want to solve is

$$\mathbb{E}_t f(y_{t+1}, y_t, y_{t-1}, u_t) = 0 \tag{24}$$

where y_t is a vector of endogenous variables and u_t, a vector of random shocks. We assume

$$
\begin{aligned}
\mathbb{E}(u_t) &= 0 \\
\mathbb{E}(u_t u_t') &= \Sigma_u \\
\mathbb{E}(u_t u_\tau') &= 0 \qquad t \neq \tau
\end{aligned}
$$

2.1 The algorithm for solving deterministic models

Deterministic, perfect foresight models, can easily be solved by making a simplifying assumption. In the type of models that we are considering, in absence of exogenous shocks, the variables converge back to the deterministic steady state asymptotically. We impose instead that they converge in finite time. If the horizon is large enough, the error of approximation introduced by this assumption is arbitrarily small, particularly at the beginning of the simulation.

With the assumption of return to the steady state at the end of the simulation, computing the trajectory of the variables is equivalent to solve a two-boundary value problem.

The model for period t can be written

$$
f(y_{t+1}, y_t, y_{t-1}, u_t) = 0 \quad . \tag{25}
$$

Models with lead and lags on more than one period can easily be brought into that format with the addition of auxiliary variables[2]

Computing the trajectory of y_τ for periods $\tau = 1, \ldots, H$ can be achieved by solving a large nonlinear problem with the equations of the model stacked for each of the H periods of the simulation:

$$
\left\{
\begin{array}{c}
f(y_0, y_1, y_2, u_1) = 0 \\
f(y_1, y_2, y_3, u_2) = 0 \\
\vdots \\
f(y_{H-1}, y_H, y_{H+1}, u_T) = 0
\end{array}
\right. \tag{26}
$$

for y_0 and y_{H+1} given, or, in more compact notation:

$$
F(Y) = 0 \tag{27}
$$

where $Y = \left[\begin{array}{cccc} y_1' & y_2' & \ldots & y_H' \end{array} \right]'$.

Laffargue (1990) shows that perfect foresight models can be solved by Newton-type methods, exploiting the particular structure of the Jacobian of $F()$ in equation (27).

In the Newton approach, the solution for vector Y is found iteratively. For an initial guess $Y^{(0)}$, successive approximated solutions $Y^{(k+1)}$ are obtained

[2] In the stochastic case, it is necessary to define auxiliary variables so as not to break nonlinear expressions and, so doing, ignore Jensen's inequality.

by solving

$$\left[\frac{\partial F}{\partial Y}\right]\left(Y^{(k+1)} - Y^{(k)}\right) = -F(Y^{(k)}) \quad ,$$

where $\frac{\partial F}{\partial Y}$ is the Jacobian matrix of function $F()$, until $||Y^{(k+1)} - Y^{(k)}|| < \epsilon_Y$ and/or $||F(Y^{(k)})|| < \epsilon_F$. Note that the effect of the exogenous shocks $u_1, \dots, u_T$ is implicitly taken into account when evaluating $F(Y^{(k)}$.

The Jacobian matrix of $F()$ has the following structure:

$$\frac{\partial F}{\partial Y} = \begin{bmatrix} B_1 & C_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & A_\tau & B_\tau & C_\tau & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & A_{H-1} & B_{H-1} & C_{H-1} \\ & & & & & A_H & B_H \end{bmatrix} \quad , \tag{28}$$

with

$$\begin{aligned} A_\tau &= \frac{\partial f(y_{\tau-1}, y_\tau, y_{\tau+1}, u_\tau)}{\partial y_{\tau-1}} \quad , \\ B_\tau &= \frac{\partial f(y_{\tau-1}, y_\tau, y_{\tau+1}, u_\tau)}{\partial y_\tau} \quad , \\ C_\tau &= \frac{\partial f(y_{\tau-1}, y_\tau, y_{\tau+1}, u_\tau)}{\partial y_{\tau+1}} \quad . \end{aligned}$$

With current computers, standard algorithms for solving sparse linear problems, such as those developed by Davis (2006), and available in Matlab of Octave, for example, can be used very efficiently in this framework.

2.2 Extended path algorithm

In order to simulate stochastic models, Fair and Taylor (1983) suggest the following approach: for each period in the stochastic simulation, draw a random vector of stochastic shocks u_t; run an auxiliary deterministic version of the model, setting shocks in period 1 to u_t and all future shocks to zero; use the value of the endogenous variables in the first period of this auxiliary model as the value of the endogenous variables in period t of the stochastic simulation.

Here is a sketch of the algorithm:

Algorithm 1 Extended path algorithm

1. $H \leftarrow$ Choose the horizon of the perfect foresight models.
2. $y_0 \leftarrow$ Choose an initial condition for the state variables.
3. **for** $t = 1$ to T **do**
4. $\quad v_t \leftarrow$ Draw independent uniform variates ($n_s \times 1$).
5. $\quad u_t \leftarrow \mathrm{P}^{-1}v$, where $P^{-1}P = Q$, the covariance matrix of shocks.
6. $\quad y_t \leftarrow$ Solve a perfect foresight model with terminal condition $y_{t+H+1} = y^\star$.
7. **end for**

In this approach, one approximates the conditional expectations by replacing the shocks by 0, their expected value. This neglects Jensen inequality and represents a stochastic simulation under a sort of certainty equivalence. Obviously, this is completely inadequate to tackle the effect of the ZLB on long run risks because the certainty equivalence always delivers a term premium that is equal to zero.

2.3 Stochastic extended path

The term premium in the model is generated by the interaction between future uncertainty and the nonlinearities of the model. It is therefore absolutely necessary to have a better treatment of the conditional expectation that translates the impact of future uncertainty on today's decisions.

We propose to compute the conditional expectation in the first K periods of the auxiliary deterministic model by quadrature methods and we will call the number of periods in which a quadrature formula is used the *order* of stochastic extended path.

There are many different quadrature methods, but all of them come down to approximate the conditional expectation by a weighted sum of outcomes of the model when it is evaluated for values of the shock in the next periods taken on a certain grid. To keep using the efficient deterministic solver, we replace again shocks by zero, but only after $K+1$ periods. So, this approach permits to take partly in consideration the joint effect of future uncertainty and nonlinearities in the model.

Each node used in the integration formula is at the origin of one or more trajectories that converge back to the steady state in period H of the auxiliary perfect foresight model. This number of trajectories taken into consideration depends upon the quadrature scheme used and increases in each period between 1 and $K+1$. We denote $J_1, \ldots, J_{K+1}$, the number of different trajectories existing in periods 1 to $K+1$. After period $K+1$, the number of trajectories remains constant and correspond to J_{K+1} parallel perfect foresight models, with shocks equal to zero but different initial values corresponding to the J_{K+1} different trajectories in period $K+1$.

If one uses an integration formula with P nodes in each period, the original model 24 is replaced by the extended perfect foresight model (EPF):

$$\sum_{i=1}^{P} \omega_i f(y_{j-1,t-1}, y_{j,t}, y_{i|j,t+1}, u_t) = 0 \qquad \text{for } t = 1, \dots, K \text{ and } j = 1, \dots, J_t \tag{29}$$

$$f(y_{j,t-1}, y_{j,t}, y_{j,t+1}, 0) = 0 \qquad \text{for } t = K+1, \dots, H \text{ and } j = 1, \dots, J_{K+1} \tag{30}$$

where $J_1 = 1$ and $J_k = (k-1)P$ for $k = 2, \dots, K+1$.

The stochastic extended path algorithm can be schematically described as:

Algorithm 2 Stochastic extended path algorithm

1. $K \leftarrow$ Choose the number of periods where the conditional expectation is computed by quadrature.
2. $[e_{i,\tau}, omega_{i,\tau}]$ Compute the nodes and weights of the integration formula.
3. $H \leftarrow$ Choose the horizon of the perfect foresight models.
4. $y_0 \leftarrow$ Choose an initial condition for the state variables.
5. **for** $t = 1$ to T **do**
6. $\quad v_t \leftarrow$ Draw independent uniform variates ($n_s \times 1$).
7. $\quad u_t \leftarrow \mathrm{P}^{-1}v$, where $P^{-1}P = Q$, the covariance matrix of shocks.
8. $\quad y_t \leftarrow$ Solve the expanded perfect foresight model with terminal condition $y_{j,t+H+1} = y^\star$.
9. **end for**

2.4 Numerical integration

When computing the expected value of a nonlinear function of a normal random variable, the quadrature formula of choice is Gauss-Hermite integration. However, this approach doesn't scale well with the number of dimensions of the integration problem and hits quickly the curse of dimensionality. It remains the preferred approach for small problems.

When the number of shocks and/or the order of stochastic extended path increases, we turn to unscented transformations (Julier, 2002).

2.4.1 Gaussian quadrature

When integrating normally distributed random variables, a natural choice is Gauss-Hermite integration. Suppose that X is a Gaussian random variable with mean zero and variance $\sigma_x^2 > 0$, and that we need to evaluate $\mathbb{E}[\varphi(X)]$, where φ is a continuous function. By definitions of the expectation and the Gaussian probability density function, we have $\mathbb{E}[\varphi(X)] = \frac{1}{\sigma_x\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)e^{-\frac{x^2}{2\sigma_x^2}}\mathrm{d}x$. This integral can be approximated using the following result (Judd, 1998):

$$\int_{-\infty}^{\infty} \phi(x)e^{-x^2}\mathrm{d}x = \sum_{i=1}^{n}\omega_i\phi(x_i) + \frac{n!\sqrt{n}}{2^n}\frac{\phi^{(2n)}(\xi)}{(2n)!}$$

for any $\xi \in \mathbb{R}$, where the last term on the right hand side is the approximation error, x_i $(i = 1, \ldots, n)$ are the roots of an order n Hermite polynomial, and the weights ω_i are positive. For a given order of approximation n, the approximation error is proportional to the order $2n$ derivative of the function to be integrated. This results tells us that is possible to find out a sequence of weights ω_i such that the evaluation of the integral with the sum of the right hand side is exact for any order $2n-1$ polynomial. Golub and Welsch (1969) describe how to calculate the quadrature weights and nodes (ω_i, x_i) by computing the eigenvalues and eigenvectors of a symmetric tridiagonal matrix. Obviously a change of variable is needed to evaluate $\mathbb{E}[\varphi(X)]$. We define $z = {}^{x}/_{\sigma_x\sqrt{2}}$, and consider the following approximation for the expectation:

$$\mathbb{E}[\varphi(X)] \approx \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \omega_i \varphi(z_i)$$

If X is a multivariate Gaussian random variable we use a Tensor product approach. For instance if X is defined in $\mathbb{R}^m$, $\mathbb{E}[X] = 0$, $\mathbb{V}[X] = \Sigma$, and $\psi(\mathbf{x})$ is a function from $\mathbb{R}^m$ to $\mathbb{R}^q$, we use the following approximation:

$$\begin{aligned}\mathbb{E}[\psi(X)] &= (2\pi)^{-\frac{m}{2}} \Sigma^{-\frac{1}{2}} \int_{\mathbb{R}^m} \psi(\mathbf{x}) e^{-\frac{1}{2}\mathbf{x}'\Sigma^{-1}\mathbf{x}} \mathrm{d}\mathbf{x} \\ &\approx \pi^{-\frac{m}{2}} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} \omega_{i_1}\omega_{i_2}\ldots\omega_{i_m} \psi(z^1_{i_1}, z^2_{i_2}, \ldots, z^m_{i_m})\end{aligned}$$

with the change of variables $\mathbf{z} \equiv (z^1, z^2, \ldots, z^q)' = \Sigma^{-\frac{1}{2}}\mathbf{x}/\sqrt{2}$. The drawback of this tensor product rule is that the number of function ψ evaluations is exponential with respect to the dimension of X.

2.4.2 Unscented transformations

When the number of shocks increases, Gauss-Hermite formula and tensor products is not practicable anymore. An alternative is to use monomial formulas (see Stroud, 1971). Recently, the theory of unscented transformations revisits this material Julier et al. (2000).

An attractive and economical approach to integrate in R^m is to use a formula with $2m+1$ nodes. In the model, the vector of exogenous random shocks u_t follows a multivariate Gaussian distribution with mean 0 and covariance matrix Σ_u. Then, the conditional expectation $\mathbb{E}_t f(y(u_{t+1}), y_t, y_{t-1}, u_t)$ can be approximated by $\sum_{i=0}^{2m} \omega_i f(y(e_i), y_t, y_{t-1}, u_t)$.

Following Julier et al. (2000) and with $P'P = \Sigma_u$, the nodes e_i, $i = 0, \ldots, 2m$ are computed as follows:

$$e_0 = 0 \tag{31}$$

$$e_{2i-1} = \sqrt{m+\alpha}P_i \qquad i = 1, \ldots, m e_{2i} \quad = -\sqrt{m+\alpha}P_i \qquad i = 1, \ldots, m \tag{32}$$

where α is an arbitrary parameter. The weights ω_i is:

$$\omega_0 = \frac{\alpha}{m+\alpha} \tag{33}$$

$$\omega_i = \frac{1}{2(m+\alpha)} \qquad i = 1,\ldots,2m \tag{34}$$

where $\sum_{i=0}^{2m} \omega_i = 1$. The unscented transformation let us recover the mean and the covariance matrix exactly for any 3rd order polynomial function. The arbitrary parameter α can be used to match another moment or characteristic of the nonlinear distribution.

2.4.3 Stochastic extended path of higher order

We call stochastic order of order K, the procedure where one computes the conditional expectations via quadrature methods during the K first periods.

If, in period 2, one wants to use again a quadrature to evaluate each node used for computing the conditional expectation in period 1, the number of nodes increases in a tree like manner and very quickly confronts again the curse of dimension.

If we consider the different problem of integrating simultaneously over m shocks and K periods, a minimal monomial approach would consider nodes that are different from zero in only one dimension at the time. This suggest to use again quadrature only for the node e_0 in the unscented transform used in the first K periods, but to reverse immediately to the deterministic model with zero shocks for all other nodes. This strategy is illustrated in Figure 2.4.3 that describes the arrangement of nodes for stochastic extended path of order 3 for a model with 2 shocks, using a 5 points unscented transformation.

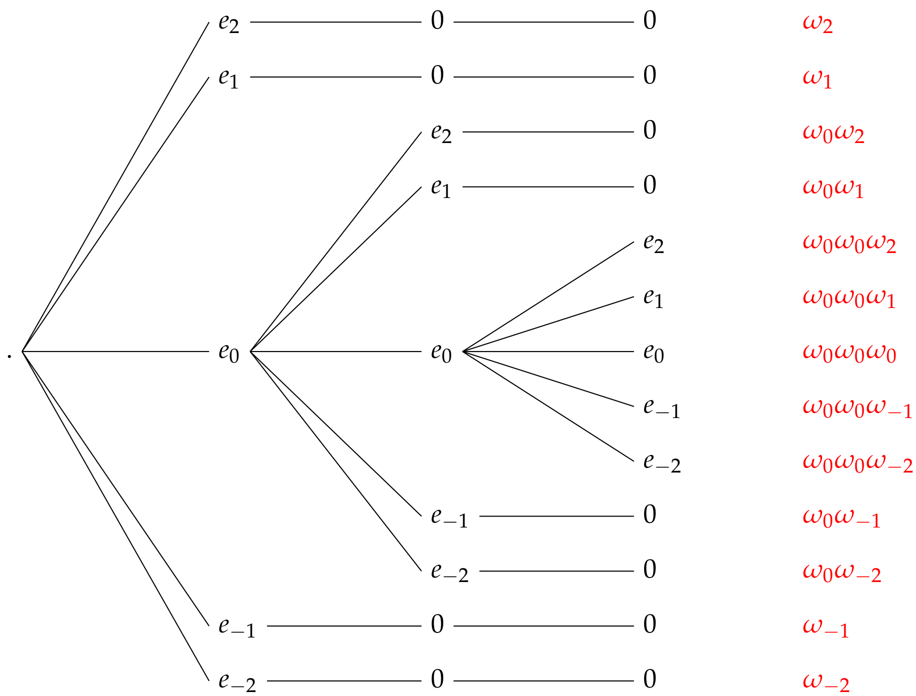

Figure 1: **Paths of future innovations in a sparse tree.** Illustration with 2 shocks, 5 nodes and order 3

This scheme corresponds to the following different trajectories:

	Trajectories												
Period	1	2	3	4	5	6	7	8	9	10	11	12	13
1	$y_{1,1}$												
2	$y_{1,2}$	$y_{2,2}$	$y_{3,2}$	$y_{4,2}$	$y_{5,2}$								
3	$y_{1,3}$	$y_{2,3}$	$y_{3,3}$	$y_{4,3}$	$y_{5,3}$	$y_{6,3}$	$y_{7,3}$	$y_{8,3}$	$y_{9,3}$				
4	$y_{1,4}$	$y_{2,4}$	$y_{3,4}$	$y_{4,4}$	$y_{5,4}$	$y_{6,4}$	$y_{7,4}$	$y_{8,4}$	$y_{9,4}$	$y_{10,4}$	$y_{11,4}$	$y_{12,4}$	$y_{13,4}$
5	$y_{1,5}$	$y_{2,5}$	$y_{3,5}$	$y_{5,5}$	$y_{5,5}$	$y_{6,5}$	$y_{7,5}$	$y_{8,5}$	$y_{9,5}$	$y_{10,5}$	$y_{11,5}$	$y_{12,5}$	$y_{13,5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H+1$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$	$\overline{y}$

.

This very large simulation exercise serves only to compute $y_{1,1}$ that is used to update the overall stochastic simulation for one period.

For those equations in periods 1 to K where a curbature formula is used to approximate the conditional expectation, the Jacobian of $\sum_{i=0}^{2m} f(y_{j+i,\tau+1}, y_{j,\tau}, y_{j,\tau-1}, u_\tau)$ is

$$\left[\begin{array}{ccc} A_{j,\tau} & B_{j,\tau} & \sum_{i=0}^{2m} \omega_i C_{j+i,\tau+1} \end{array} \right] \quad . \tag{35}$$

Despite the fact that the conditional expectation is in front of the nonlinear function, in the Jacobian matrix, the curbature formula concerns only the Jacobian with respect to the forward looking variables.

2.5 Hybrid stochastic extended path

It would take a very high order of stochastic extended path to fully take into account the effects of future volatility. Furthermore, the deterministic steady state used as terminal value in the auxiliary deterministic, perfect foresight, model isn't a steady state anymore when one takes into account the effects of future volatility.

On the other hand, the perturbation approach provides us with another approximation of the effects of future volatility. The idea behind *hybrid stochastic extended path* is to combine the treatments of major nonlinearities in the model with the stochastic extended path method with taking into accounts general effects of future volatility with a perturbation approach.

Let's introduce σ, the stochastic scale of the model, such that

$$u_{t+1} = \sigma \epsilon_t \quad , \tag{36}$$

where ϵ_t is a vector of m auxiliary random variables with zero mean and covariance matrix Σ_ϵ. It follows that $\Sigma_u = \sigma^2 \Sigma_\epsilon$.

Consider the vector of solution functions $g()$ such that

$$y_t = g(y_{t-1}, u_t, \sigma) \quad ,$$

and the original model (24) is satisfied. Functions $g()$ are unknown.

Plugging the postulated functions $g()$ in (24), one obtains

$$\mathbb{E}_t F(y_{t-1}, u_t, \epsilon_{t+1}, \sigma) = \mathbb{E}_t f(g(g(y_{t-1}, u_t, \sigma), \sigma\epsilon_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t) = 0 \quad . \tag{37}$$

In the above expression, the only stochastic term from the point of view of the conditional expectation is ϵ_{t+1}.

The *hybrid stochastic extend path* approach considers an Taylor expansion in the sole direction of σ:

$$\begin{aligned}
&\mathbb{E}_t f(g(g(y_{t-1}, u_t, \sigma), \sigma\epsilon_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t) \\
&\quad = f(g(g(y_{t-1}, u_t, 0), 0, 0), g(y_{t-1}, u_t, 0), y_{t-1}, u_t) + \mathbb{E}_t \sum_{i=1}^{\infty} \frac{1}{i!} \frac{\partial^i F}{\partial \sigma^i} \sigma^i \quad .
\end{aligned} \tag{38}$$

In the first K periods of the stochastic extended path approach, the quadrature takes into account jointly the deterministic aspects and the effects of future volatility. After the first K periods, the deterministic solution setting shocks to zero corresponds to the first term on the right hand side of (38). Using a perturbation in σ direction would help taking into account the effects of future volatility after the first K periods.

In practice, considering a 2nd order perturbation, the terms in periods $K+1$, entering into the quadrature formula for period K, would be corrected in the following manner:

$$\widetilde{y}_{t+K+1} = y_{t+K+1} + \frac{1}{2} g_{\sigma^2} \quad , \tag{39}$$

where y_{t+k+1} is the value computed by the deterministic simulation approach and g_{σ^2} is the second derivative of the solution function with respect to the stochastic scale of the model.

3 The effect of the ZLB on the term premium

In order to assess the effects of the zero lower bound for nominal interest rates, we run a stochastic simulation of the Rudebusch and Swanson (2008) model, considering only shocks to total factor productivity, A_t. In this baseline simulation we use hybrid stochastic extended path of order 5 (using quadrature for the 5 first periods in the auxiliary model).

It happens that a succession of positive shocks depresses so much the interest rate that the nominal interest rate, in absence of the zero lower bound for nominal interest rate would become negative. The evolution of nominal interest rate, with and without the ZLB is represented in Figure 3. Note that this scenario is very different from what we observed during the recent crisis that were caused, on the contrary, by demand and confidence shocks.

In Figure 3 we plot the term premium with and without the ZLB. As expected, as in Rudebusch and Swanson (2008) the ZLB puts a floor below the nominal interest rate and the term premium with the ZLB is lower than without during the episode during which the unconstrained nominal interest rate would become negative.

Comparing the evolution of consumption (Figure 3) and the evolution of the bond nominal yield (Figure 3), one observes that the ZLB has a relatively stronger effect on yield rather than on consumption.

Several parameters need to be tuned when implementing hybrid stochastic extended path. As a robustness check, we vary the order of hybrid stochastic extended path between 1 and 15. In Figure 3, we plot the evolution of the term premium, in absence of ZLB for order=1,2,3,4,5,10 and 15.

The difference is important between orders 1 and 4, but much less between 4 and 15. Depending whether one uses order 1 or order 4, the overall magnitude of the term premium almost doubles. On the other hand, using more than order 5, doesn't seem to bring significant additional information.

4 Conclusion

We present a hybrid stochastic extended path method that lets us analyze the effects of the ZLB on asset prices in a new-Keynesian model, a question difficult to address with usual methods.

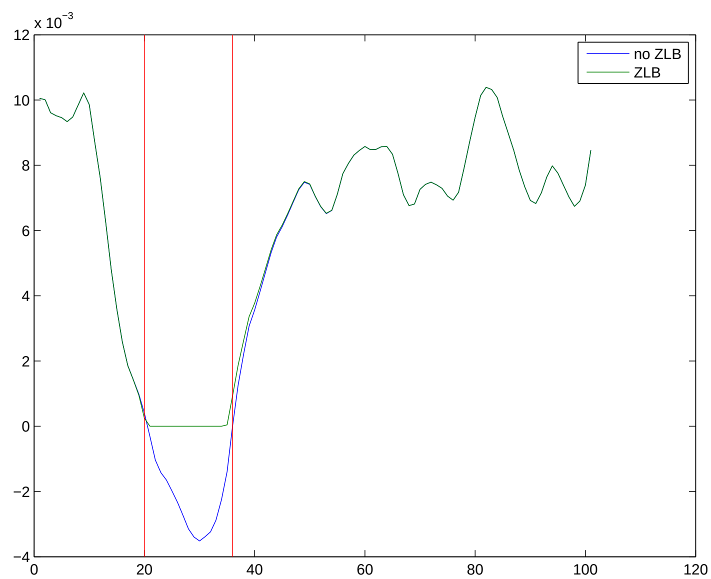

Figure 2: Nominal interest rate, with and without the ZLB

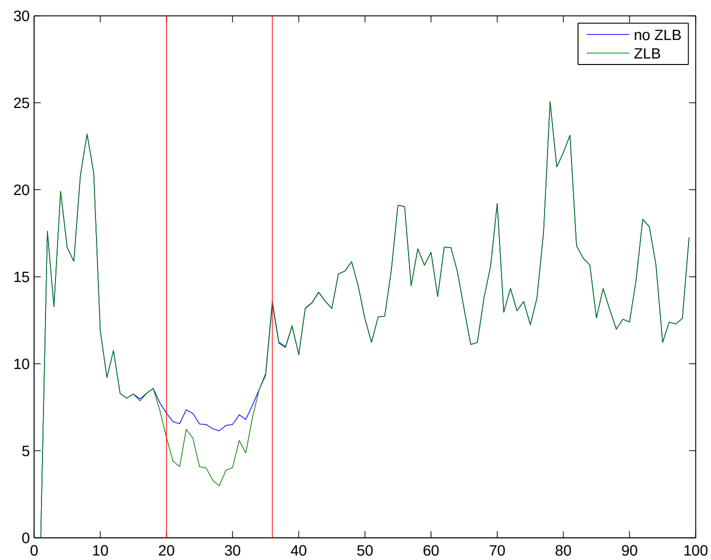

Figure 3: Term premium, with and without the ZLB

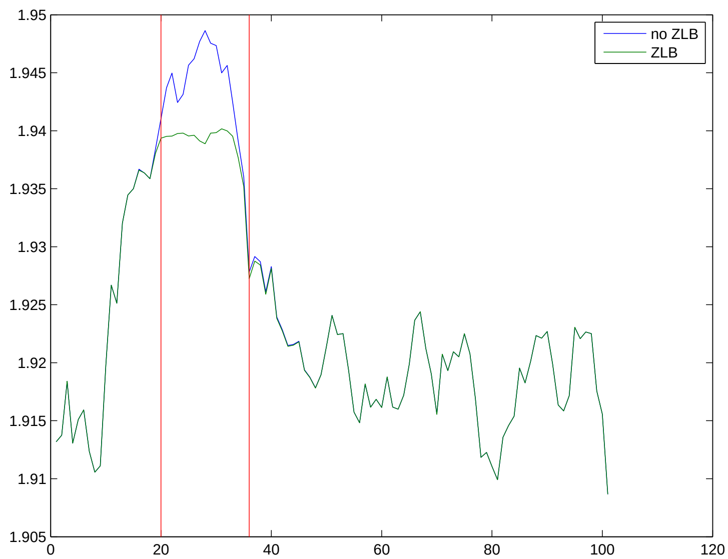

Figure 4: Real consumption, with and without the ZLB

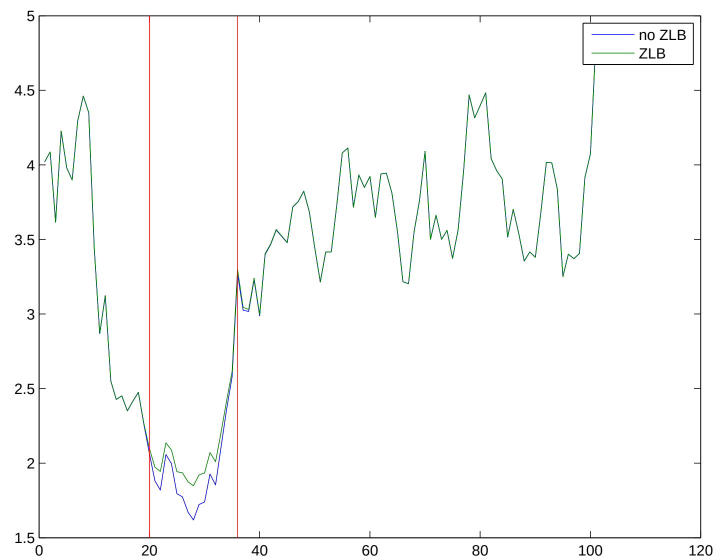

Figure 5: Bond yield in percentage, with and without the ZLB

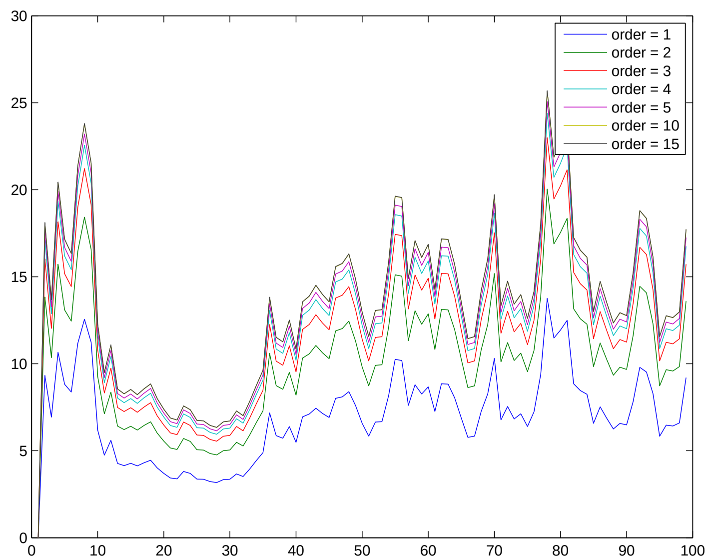

Figure 6: Term premium, without the ZLB, various orders

In our example, the term premium is maintained higher than otherwise by the ZLB that limits the fluctuation of the interest rate.

The approach seems promising. It still needs to include other shocks in addition to the technological shock, in order to address scenarios more similar to the evolutions during the Great Recession. It is also necessary to get a better sense of the accuracy obtained with this method.

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