ACCURACY OF THE EXTENDED PATH SIMULATION METHOD IN A NEW KEYNESIAN MODEL WITH ZERO LOWER BOUND ON THE NOMINAL INTEREST RATE

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ABSTRACT. In this paper we evaluate the accuracy of the Extended Path approach for solving DSGE models with occasionally binding constraints. We consider a New Keynesian model with Calvo price setting, an aggregation technology of intermediary goods *à la* Kimball, to control for the degree of nonlinearity in the model, and a Zero Lower Bound on the nominal interest rate. Accuracy errors show to be quite reasonable but deteriorates significantly when the ZLB is binding.

1. Introduction

The aim of this paper is to evaluate the accuracy of the extended path approach when solving a non linear model with occasionally binding constraints.

In a previous contribution, *i.e.* Adjemian and Juillard (2010), we showed that using the extended path method, first proposed by Fair and Taylor (1983), it was indeed possible to estimate, by the Simulated Method of Moments, a DSGE model including a zero lower bound for the nominal interest rate (or more generally a DSGE model including occasionally binding constraints) .

The extended path approach relies on a perfect foresight solver to take full account of the non–linearities introduced by the occasionally binding constraints. For each period of the sample, exogenous innovations are treated as surprise shocks in the period of a deterministic simulation where shocks are set to their expected value of zero, in all future periods. This approach neglects Jensen inequality, but we considered that it was a minor drawback in comparison with the correct treatment of the non–linearities induced by the zero lower bound.

Very few studies, *e.g.* Gagnon (1990) and Love (2009), evaluate the accuracy of this simulation method. These authors, considering a stochastic growth model, show that the approximation errors are reasonable and that the extended path approach performs as well (or even better) as a global approximation approach. However the degree of non linearity of the stochastic growth model with a Cobb-Douglas technology is relatively weak (for credible values of the deep parameters), so that one may think that their results are not fully convincing. Intuition suggests that the magnitude of the errors induced by neglecting the Jensen inequality will depend on the degree of nonlinearity in the forward terms appearing in the original model.

We propose to build a New-Keynesian model with Calvo (1983) nominal rigidity on prices, Kimball (1996) aggregation function of intermediate goods and a zero lower bound on nominal

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interest rates. Considering this model we will evaluate the accuracy of the extended path approach, controlling the degree of non linearity with the deep parameter associated to the Kimball (1996) aggregation function[1]. As the true decision rules are unknown with the extended path method, it is not possible to evaluate approximations errors directly as in, for instance, Juillard and Villemot (2010). It is however possible to compute residuals of the original equations of the model in which we plug the solution paths from the extended path method. These residuals deliver an estimation of the error of approximation. We compute such approximation errors for state points on an hyper-sphere centered on the deterministic steady state and along simulated paths.

In section 2 we present the New-Keynesian model considered to evaluate the accuracy of the extended path approach. The simulation method is presented in section 3 and the accuracy checks are introduced in section 4. Section 5 reports the results and concludes.

2. Description of the model

In order to keep the model as simple as possible, we consider an economy without capital accumulation and the unique imperfection is related to the production sector.

2.1. **Households.** We consider an economy populated by a continuum of infinitely living identical households. Each household values the consumption of an homogeneous good and leisure. The inter-temporal utility function of the representative household is given by:

$$\mathscr{W}_t = \mathscr{U}(C_t, h_t) + \beta \mathbb{E}_t \left[\mathscr{W}_{t+1}\right] \tag{H1}$$

with

$$\mathscr{U}(C_t, h_t) = \frac{C_t^{1-\sigma_C}}{1-\sigma_C} - \xi_h \frac{h_t^{1+\eta}}{1+\eta} \tag{H2}$$

where C_t is the level of consumption at time t, h_t is the supply of labor (hours), $\beta \in [0,1]$ is the discount factor, $\sigma_C > 0$ is the intertemporal elasticity of consumption and $\eta > 0$ the Frisch elasticity of labor.

The budget constraint of the representative household, in period t, is the following:

$$P_t C_t + \frac{B_t^d}{\varepsilon_{B,t} R_t} = B_{t-1}^d + W_t h_t + \mathscr{D}_t + T_t \tag{H3}$$

where P_t is the aggregate price index; $R_t = 1+i_t$, corresponds to the rate of interest plus one; B_t^d the nominal value of bonds detained at the end of period t; $\varepsilon_{B,t}$ is the risk premium requested by households in order to detain the bond, $\varepsilon_{B,t}$ is an exogenous stochastic process with asymptotic expectation equal to one; W_t is the hourly wage rate received in period t; T_t represents net transfers received by the household during the period; $\mathscr{D}_t(h)$ are the nominal dividends received from firms in the intermediate good sector.

The representative household chooses its consumption, labor supply, bond holdings so as to maximize its inter-temporal utility (H1)-(H2) under the budget constraint (H3), taking as given the distribution of prices and exogenous variables.

[1] We could also control the degree of nonlinearity by varying the inter-temporal elasticity of substitution of the representative household. But it is very likely that such variations would not affect the accuracy because there is no capital accumulation in this model.

2.2. Production.

2.2.1. *Final good.* The final good, used for private consumption, is obtained by aggregating a continuum of intermediate goods $z \in [0,1]$ in a perfectly competitive sector. The representative firm use a constant return to scale technology *à la* Kimball (1996):

$$\int_0^1 \mathscr{G}\left(\frac{Y_t(z)}{Y_t}\right) dz = 1 \tag{FGS1}$$

where $\mathscr{G}$ is a strictly increasing concave function such that $\mathscr{G}(1) = 1$. We follow Dotsey and King (2005) or Levin et al. (2007) and adopt the following functional form for this aggregation function:

$$\mathscr{G}(x) = \frac{\phi}{1+\psi}\left[(1+\psi)x - \psi\right]^{\frac{1}{\phi}} - \left[\frac{\phi}{1+\psi} - 1\right] \tag{FGS2}$$

where $\phi = \frac{\varepsilon(1+\psi)}{\varepsilon(1+\psi)-1}$, $\psi \leq 0$ and $\varepsilon > 0$. This technology of aggregation is a generalization of the well known Dixit and Stiglitz (1977) CES aggregation technology, one can check that (FGS1) and (FGS2) reduces to the standard Dixit and Stiglitz (1977) aggregation function when $\psi = 0$. This parameter controls for the curvature of the demand function, the more ψ is negative the more kinked is the demand function (see Levin et al. (2007)).

Given relative prices, the representative firm maximizes its profit subject to the technological constraint (FGS1) and (FGS2):

$$\begin{aligned} &\max_{\{Y_t(z);z\in[0,1]\}} Y_t - \int_0^1 \frac{P_t(z)}{P_t} Y_t(z)\mathrm{d}z \\ &\qquad\qquad \underline{\text{sc}} \quad (FGS1) - (FGS2) \end{aligned} \tag{FGS3}$$

the solution of this program defines the demand for each intermediate good as a function of relative prices, and the zero profit condition defines the aggregate price P_t.

2.2.2. *Intermediary goods.* There is a continuum of intermediate goods indexed by $z \in [0,1]$. Each good z is produced by a unique firm z using labor. The production technology is linear in the labor input. That is:

$$Y_t(z) = A_t l_t(z) \tag{IGS1}$$

$Y_t(z)$ is the amount of good produced by firm z, $l_t(z)$ is the amount hours used in the production process and A_t is an exogenous stochastic process with asymptotic mean equal to $A^\star$.

It can be shown that the marginal cost is invariant in the cross section of firms, we denote $mc_t = {}^{w_t}/_{A_t}$ the real marginal cost. The nominal profit of an intermediate firm that offers price $\mathcal{P}$ at date t is given by:

$$\Pi_t(\mathcal{P}) = \left(\frac{\mathcal{P}}{P_t} - mc_t\right) P_t Y_t(z)$$

Due to the monopolistic competition between firms producing imperfectly substitutable intermediate goods, each firm z has market power. Nevertheless, a firm can't decide of its optimal price in each period. Following a Calvo (1983) scheme, at each date, the firm receives a signal telling it whether it can revise its price $P_t(z)$ in an optimal manner or not. There is a probability ν that the firm can't revise its price in a given period. In such a case, the firm follows the following rule:

$$P_t(z) = \pi^\star P_{t-1}(z)$$

where $\pi^\star$ is the inflation target of the monetary authorities. When a firm z receives a positive signal (with probability $1-\nu$), it chooses price $P_t(z)$ that maximizes its profit knowing that in the next periods she may not be able to choose optimally its price.

2.3. **Government and monetary authority.** The government issues lump sum monetary transfers, T_t, to the households and bounds. The government budget constraint is given by:

$$\frac{B_t^s}{\varepsilon_{B,t}R_t} - B_{t-1}^s - T_t = 0$$

where R_t is the nominal interest factor set by the monetary authorities.

It is assumed that a central bank controls the nominal interest rate according to the following rule:

$$R_t = \max\left\{1, R^\star\left(\frac{\pi_t}{\pi^\star}\right)^{r_\pi}\right\}$$

where $\pi^\star$ is the inflation target of the central bank. The max function constrains the nominal interest factor to be greater or equal than one. This equation defines two regimes. In the first one, the nominal interest rate is constant (equal to zero), whereas in the second one the nominal interest rate reacts to the current excess inflation. In the sequel, we consider the case where the steady state of the economy is in the second regime, *ie* the long run level of the nominal interest rate is assumed to be strictly positive. Nevertheless, during the transitions to this steady state the economy can hit the Zero Lower Bound for the nominal interest rate. Note that we omit the traditional output gap variable and the lagged interest rate. Also, the nominal interest rate depends on the current inflation in deviation to the target and not the lagged inflation in deviation to the target (as in Smets and Wouters (2007)). Adopting a more traditional Taylor rule would add two state variables (lagged inflation and interest factor) without any additional advantage with respect to our problem.

2.4. **Equilibrium.**

2.4.1. *Price distortion.* Prices in the intermediary good sector are heterogeneous. However, this heterogeneity doesn't hinder aggregation because the technology in the sector of intermediary goods is linear. Defining $l_t = \int_0^1 l_t(z)dz$ as the aggregate demand for labor and $\mathscr{Y}_t = \int_0^1 Y_t(z)dz$ the sum of intermediary productions, we have directly:

$$\mathscr{Y}_t = A_t l_t$$

Because the aggregation technology (FGS1)- (FGS2) is not linear, the sum of intermediary productions is different from Y_t, the demand of final good. Integrating the demand function for good z from the final good producer, *ie* the solution of (FGS3), over z, we get:

$$\mathscr{Y}_t = \Delta_t Y_t$$

with the distortion Δ_t is defined as:

$$\Delta_t \equiv \frac{1}{1+\psi}\int_0^1 \left(\left(\frac{P_t(z)/P_t}{\Sigma_t}\right)^{-(1+\psi)\varepsilon} + \psi\right)\mathrm{d}z$$

where Σ_t is the Lagrange multiplier associated to the program (FGS3) of the representative final good producer.

2.4.2. *Dividends paid by intermediary good firms.* Firms in the intermediary good sector interact in monopolistic competition and make profits that are paid to households in the form of dividends. The sum of nominal profits at date t is

$$\Pi_t = \int_0^1 \Pi_t(z)\mathrm{d}z = \int_0^1 \left(\frac{P_t(z)}{P_t} - mc_t\right) P_t Y_t(z) = P_t\left(Y_t - w_t l_t\right)$$

As households own the firms, the profits are paid to them. The repartition of these profits between the households is undetermined in general equilibrium, but we know that

$$\int_0^1 \mathscr{D}_{1,t}(h)\mathrm{d}h = P_t\left(Y_t - w_t l_t\right) \tag{IGS2}$$

2.4.3. *Equilibrium in labor and bond markets.* In general equilibrium, labor supply from the representative household equals aggregate labor demand by firms of the intermediary good market and the demand and supply of bonds are equal:

$$h_t = l_t, \quad B_t^d = B_t^s$$

2.4.4. *Equilibrium on the final good market.* By substituting the equilibrium condition on the bond market, the definition of aggregate dividends and the budget constraint of the government in the budget constraint of the representative household, we get:

$$P_t C_t = W_t h_t + P_t\left(Y_t - w_t l_t\right)$$

Knowing that the labor market is in equilibrium, we obtain:

$$C_t = Y_t = \Delta_t^{-1}\mathscr{Y}_t$$

2.5. **Equations of the model.** The evolution of the endogenous variables satisfies the following set of stochastic difference equations:

$$\begin{cases} A_t &= A^\star e^{a_t - \frac{1}{2}\frac{\sigma_a^2}{1-\rho_a^2}} \\ a_t &= \rho_a a_{t-1} + u_{a,t} \end{cases} \tag{1}$$

$$\begin{cases} \varepsilon_{B,t} &= e^{b_t - \frac{1}{2}\frac{\sigma_b^2}{1-\rho_b^2}} \\ b_t &= \rho_b b_{t-1} + u_{b,t} \end{cases} \tag{2}$$

$$\lambda_t = C_t^{-\sigma_c} \tag{3}$$

$$\xi_h h_t^\eta + \lambda_t w_t = 0 \tag{4}$$

$$\beta\varepsilon_{B,t}\mathbb{E}_t\mathscr{E}_{1,t+1} - \frac{\lambda_t}{R_t} = 0 \tag{5}$$

$$-\mathscr{Z}_{1,t} + \frac{w_t}{A_t}\Theta_t^{-\phi}Y_t + \nu\beta\frac{\mathbb{E}_t\mathscr{E}_{2,t+1}}{\lambda_t} = 0 \tag{6}$$

$$-\mathscr{Z}_{2,t} + \Theta_t^{-\phi}Y_t + \nu\beta\frac{\mathbb{E}_t\mathscr{E}_{3,t+1}}{\lambda_t} = 0 \tag{7}$$

$$-\mathscr{Z}_{3,t} + Y_t + \nu\beta\frac{\mathbb{E}_t\mathscr{E}_{4,t+1}}{\lambda_t} = 0 \tag{8}$$

$$\frac{\mathscr{Z}_{2,t}}{(1+\psi)(1-\phi)}p_t^{\frac{\phi}{1-\phi}} + \mathscr{Z}_{3,t}\frac{\psi}{1+\psi} + \mathscr{Z}_{1,t}\frac{\phi}{(1+\psi)(\phi-1)}p_t^{\frac{\phi}{1-\phi}-1} = 0 \tag{9}$$

$$-\Theta_t + (1-\nu)p_t^{\frac{1}{1-\phi}} + \nu\left(\frac{\pi^\star}{\pi_t}\right)^{\frac{1}{1-\phi}}\Theta_{t-1} = 0 \tag{10}$$

$$-\left((1+\psi) - \Theta_t^{1-\phi}\right) + (1-\nu)\psi p_t + \nu\frac{\pi^\star}{\pi_t}\left(1+\psi-\Theta_{t-1}^{1-\phi}\right) = 0 \tag{11}$$

$$-\Delta_t + (1-\nu)p_t^{\frac{\phi}{1-\phi}} + \nu\left(\frac{\pi^\star}{\pi_t}\right)^{\frac{\phi}{1-\phi}}\Delta_{t-1} = 0 \tag{12}$$

$$\frac{\Theta_t^{-\phi}\Delta_t + \psi}{1+\psi}Y_t - A_t h_t = 0 \tag{13}$$

$$-Y_t + C_t = 0 \tag{14}$$

$$R_t - \max\left\{1, R^\star\left(\frac{\pi_t}{\pi^\star}\right)^{r_\pi}\right\} = 0 \tag{15}$$

$$\mathscr{E}_{1,t} - \frac{\lambda_t}{\pi_t} = 0 \tag{16}$$

$$\mathscr{E}_{2,t} - \lambda_t\left(\frac{\pi^\star}{\pi_t}\right)^{\frac{\phi}{1-\phi}}\mathscr{Z}_{1,t} = 0 \tag{17}$$

$$\mathscr{E}_{3,t} - \lambda_t\left(\frac{\pi^\star}{\pi_t}\right)^{\frac{1}{1-\phi}}\mathscr{Z}_{2,t} = 0 \tag{18}$$

$$\mathscr{E}_{4,t} - \lambda_t\frac{\pi^\star}{\pi_t}\mathscr{Z}_{3,t} = 0 \tag{19}$$

where $\mathbb{E}_t[\mathscr{X}] = \mathbb{E}_t[\mathscr{X}|\Omega_t]$ is the expectation of $\mathscr{X}$ conditional on the information at time t, Ω_t. The information available to the agents at time t is the present and past realizations of the variables, $\Omega_t = \{(s_\tau, u_\tau, y_{\tau-1}, x_{\tau-1}, \mathscr{E}_\tau); \forall\tau \leq t\}$[2]. Equations (1)-(2) define the law of motion of the exogenous stochastic processes, the productivity and risk premium shocks. By construction, these equations are such that the ergodic expectations of the exogenous variables are respectively $A^\star$ and 1. Equation (3) defines the Lagrange multiplier associated to the budget constraint of the representative household (H3) as the marginal utility of consumption. Equation (4) defines the optimal trade-off between consumption and leisure. Equations (5) and (16) form the Euler equation associated to the demand of bonds from the representative household. Equations (6) to (12) are derived from the final and intermediate sector firms optimal behavior and characterizes the dynamic of inflation (see appendix A). Equation (13) relates the aggregate demand with the aggregate supply, equation (14) is the resource constraint. Equation (14) is the Taylor rule. Finally, equations (16) to (19) define four auxiliary variables, the non linear combinations of endogenous variables appearing as expected terms in the previous equations.

[2]Although, this is not transparent in our notation, the information set also includes the model and its parameters, the agents in the model know the data generating process.

3. Extended path approach

The previous model, described by equations (1) to (19), may be represented more generally as follows:

$$s_t = \mathsf{Q}(s_{t-1}, u_t) \tag{20a}$$

$$\mathbb{E}_t\left[\mathsf{F}(y_t, x_t, x_{t-1}, s_t, \mathscr{E}_{t+1})\right] = 0 \tag{20b}$$

$$\mathsf{G}(y_t, x_t, x_{t-1}, s_t) = 0 \tag{20c}$$

$$\mathscr{E}_t = \mathsf{H}(y_t, x_{t-1}, s_t) \tag{20d}$$

where s_t is a $n_s \times 1$ vector of exogenous variables (productivity and risk premium), the innovation u_t is a multivariate random variable in $\mathbb{R}^{n_s}$ with expectation 0 and variance Σ_u (the cumulative distribution function of u is denoted $\mathrm{P}(u)$), x_t is a $n_x \times 1$ vector of endogenous state variables (the distortion related to price rigidity, Δ_t, and the Lagrange multiplier associated to aggregation technology in the final good sector, Θ_t), y_t is a $n_y \times 1$ vector of non predetermined variables (C_t,λ_t, w_t, π_t, R_t, $\mathscr{Z}_{1,t}$, $\mathscr{Z}_{2,t}$, $\mathscr{Z}_{3,t}$, Y_t, p_t and h_t) and $\mathscr{E}_t$ is a $n_{\mathscr{E}} \times 1$ vector of auxiliary variables. Q, F, G and H are non linear continuous functions (not necessarily differentiable everywhere) respectively representing equations (1)-(2), equations (5) to (8), equations (3)-(4) and (9) to (15) and equations (16) to (19).

3.1. EP algorithm.

The extended path algorithm (EP hereafter) is a simulation approach for generating time series for the endogenous variables $\{y_t, x_t\}_{t=1}^T$ given an initial condition for the state variables, (s_0, x_0), and a sequence of innovations $\{u_t\}_{t=1}^T$. The extended path approach indirectly characterizes the decision rules (*ie* the functions relating the non predetermined variables, y_t, with the state variables, x_{t-1} and s_t) by generating time-series for the endogenous variables. Basically, the trick is to extend the information set at date t in the following way:

$$\widetilde{\Omega}_t = \Omega_t \cup \{u_\tau = 0; \forall \tau > t\}$$

Given the state of the economy at date t, (x_{t-1}, s_t), we can then solve (20)[3] for y_t by solving a perfect foresight model (as described in Laffargue (1990)). Here is a sketch of the algorithm:

Algorithm 1 Extended path algorithm

1. $H \leftarrow$ Set the horizon of the perfect foresight models.
2. $(x_0, s_1) \leftarrow$ Choose an initial condition for the state variables.
3. **for** $t = 1$ to T **do**
4. $\quad (y_t, z_t) \leftarrow$ Solve a perfect foresight model with terminal condition $y_{t+H} = y^\star$.
5. $\quad v \leftarrow$ Draw independent uniform variates $(n_s \times 1)$.
6. $\quad u \leftarrow \mathrm{P}^{-1}(v)$
7. $\quad s_{t+1} \leftarrow \mathsf{Q}(s_t, u)$
8. **end for**

The main approximation here is that we assume that the agents believe that the innovations of the exogenous states will be zero in the future. There is no uncertainty about the future.

[3] With expectations conditional on $\widetilde{\Omega}_t$ and not Ω_t.

3.2. **Remarks.** The main advantage of this approach is that we can simulate large models *with an arbitrary precision*, because the number of needed operations increases polynomially with the number of endogenous variables (the main task when solving the perfect foresight model consist in inverting a sparse matrix) and not exponentially (as it would with a global approximation of the policy rules). The extended path approach does not suffer from the so called curse of dimensionality. A second advantage is that the EP approach does not require any special treatment when the model admit occasionally binding constraints, because it does not impose the differentiability of F or G.

Obviously these advantages come at a cost: with the EP approach we abstract from the effects of uncertainty on the behavior of the agents. However, two points are worth noting. First, large scaled models are usually solved considering a first order Taylor approximation of Q, F, G and H in (20). If we linearize the model, we will also neglect the effects of uncertainty about the future *and* we will not be able to treat the occasionally binding constraints. In this respect the EP approach dominates the first order perturbation approach. We could instead consider a k-order perturbation approach. If k is greater than one, the certainty equivalence property is not satisfied and uncertainty about the future has an impact on agents decisions. Nevertheless, because this approach requires the differentiability everywhere of the model, we would not be able to treat correctly occasionally binding constraints. Second, if we agree with Lucas (1987, 2003) that the cost of fluctuations is very small, it is most likely that the error of approximation induced by the substitution of Ω_t by $\widetilde{\Omega}_t$ is small.

3.3. **Numerical illustration.** We illustrate the EP method by comparing different simulated paths obtained with different approaches: EP, first and second (with or without pruning) order perturbations. The calibration of the deep parameters is described in table 1. We simulate the model, as defined by equations (1) to (19), considering the deterministic steady state as an initial condition and for all the approaches we use the same sequence of (Gaussian) structural innovations. Results are presented in figures 1, 2 and 3. The first remarkable feature is that the second order perturbation simulated paths (either with our without pruning) are closer to the EP simulated paths than the first order perturbation simulated paths. When the Zero Lower Bound is not binding[4] (at current time t or in the expectations of the agents) the second order perturbation simulated output is, on average, higher than the EP simulated output (0.017% in terms of steady state level of output). Under the same conditions, the first order perturbation simulated output is, on average, higher than the EP simulated output by 0.021%. One can conclude that the discrepancies between EP and perturbation approaches are fairly small when the ZLB is not an issue. Figure 1 shows that this conclusion does not hold when the ZLB is binding. In this situation the differences between EP and perturbation approaches are much more important, up to 8% in terms of steady state level of output. Put differently, if the EP algorithm provides an accurate solution[5], it means that using a perturbation approach to compute forecasts or IRFs we may overestimate the level of output or consumption by 8% when the Zero Lower Bound is binding (or when the agents expect that the ZLB will bind).

Figures 4 and 5 plot the distribution of the state variables, when considering the EP approach. In figure 4, we clearly see that, with this model and its baseline calibration described in table 1, the ZLB binds in presence of large deflationary efficiency shocks. Overall, the probability of hitting the ZLB is around 1%. The volatility of the endogenous state variables is generally pretty small.

[4] Or when the nominal interest rate is not negative if we do not impose the ZLB constraint, as in figure 2.

[5] We still have to establish this point. We turn to the accuracy issue in the section 5.

Larger deviations from the steady state are only observed when the nominal interest rate hits the ZLB. We would not observe such deviations if the paths were simulated with a perturbation approach[6].

In the next section we present, in a general framework, two approaches for evaluating the accuracy of the EP method with respect to the uncertainty about the future.

4. Accuracy checks

Using the EP approach to solve (20), we can, in principle, perfectly control the accuracy of the solution with respect to the deterministic equations (20a), (20c) and (20d). Consequently, we only have to check the accuracy of the solution with respect to the Euler type equations (20b), which can be rewritten as a multivariate integral:

$$\mathscr{R}(x_-,s) \triangleq \int_\Lambda \mathsf{F}\left(y,x,x_-,s,\mathsf{H}\left(y_+(u),x,\mathsf{Q}(s,u)\right)\right)\mathrm{dP}(u) = 0 \quad \forall\,(x_-,s)\in\Xi\subseteq\mathbb{R}_+^{n_x+n_s}$$

where $\Lambda\subseteq\mathbb{R}^{n_s}$ and $(y,y_+(u),x)$ is provided by the EP algorithm 1 given initial conditions (x_-,s).

4.1. Accuracy on a growing sphere (Test #1).

The first test consist in evaluating the residual $\mathscr{R}(x_-,s)$ with state variables (x_-,s) uniformly distributed on an hyper-sphere centered on the deterministic steady state, $(x^\star,s^\star)$. In our case, the two endogenous states are Θ_t and Δ_t and we have $(x^\star,s^\star)=(1,1,A^\star,1)$ (see appendix B). Obviously, we cannot evaluate analytically the multivariate integral appearing in the definition of the residuals. We use a Gaussian quadrature approach (based on Hermite orthogonal polynomials because we assume that the innovations are Gaussian, see Abramovitz and Stegun (1964, chapters 22 and 25.4)) to approximate these residuals. In the tables of results we report the max of the approximated residuals[7][8],

$$\max_{(x_-,s)\in\mathscr{S}_r}\left|\widehat{\mathscr{R}}(x_-,s)\right|$$

where $\mathscr{S}_r=\{(x_-,s)\in\mathbb{R}_+^{n_x+n_s};||(x_-,s)-(x^\star,s^\star)||_2=r\}$ is the hyper-sphere centered on the deterministic steady state with radius $r>0$. We use a Quasi Monte-Carlo approach to generate points uniformly distributed in $\mathscr{S}_r$ (see Sobol (1977), Antonov and Saleev (1979), Bratley and Fox (1988) and Joe and Kuo (2003)).

4.2. Accuracy along a simulated path (Test #2).

The previous test has two shortcomings. First, considering a uniform distribution over an hyper-sphere is inefficient because the ergodic distribution of the endogenous variables is not uniform. Consequently, we compute too much residuals in regions of the state space where the ergodic probability mass implied by the structural model is near zero. Second, this test ignores how the approximations errors are cumulated. Test # 2 consist in simulating paths for the endogenous variables with EP algorithm 1. On each point of the stochastic simulation we compute the approximated residuals $\widehat{\mathscr{R}}(x_-,s)$, and we report various moments (max, mean,...).

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[6]Note that with a first order perturbation approximation these two variables are constant, the endogenous state variables vanish. Note also that in this model, the steady state levels of the state variables are on the boundary of the admissible values for these variables (price distortion, Δ_t, is greater or equal to one, and Ω_t is between zero and one). This can be problematic because we are not able here to define the Taylor approximation on a open interval (and it is not clear if k-order approximations will deliver paths such that $\Delta_t\geq 1$ and $0\leq\Omega_t\leq 1$).

[7]Note that F is a vector of Euler type equations, in the tables of results we report the statistics for each Euler equation.

[8]In the tables 2 to 5 and in figures 6(a) to 6(d), residuals of equations (5) to (8) are normalized so as to be expressed as ratio of C_t, $\mathscr{Z}_{1,t}$, $\mathscr{Z}_{2,t}$ and $\mathscr{Z}_{3,t}$, respectively.

5. Results and final comments

5.1. **Preliminaries.** Before the presentation and discussion of our results, two remaining issues specific to our model need to be stressed. First, as illustrated in figure 5, the endogenous state variables are not defined around the deterministic steady state. The price distortion, Δ, can take values equal to its steady state or above, *ie* $\Delta \geq 1$ and the transformed Lagrangian satisfies $0 \leq \Theta \leq \Theta^{\star} = 1$. Consequently, it would not make any sense to consider values of the endogenous states uniformly distributed on the (whole) hyper-sphere. Doing so, we would obtain endogenous states, x_{-}, such that $\Delta_{-} < 1$ and/or $\Theta_{-} > 1$. Even if our experience show that we are able to solve the model under these conditions, the results are then meaningless. We need to restrict the uniform distribution on a region of the hyper-sphere. To do so we simply reflect the points against the steady state when the levels of the endogenous states are meaningless. That is, if for a point $(x_{-}, s) = (\Delta_{-}, \Theta_{-}, A, \varepsilon_B) \in \mathscr{S}_r$ we have, say, $\Delta_{-} < 1$, we redefine the price distortion, Δ_{-}, as $1 + 1 - \Delta_{-}$. Second, in the sequel we will evaluate the accuracy of the EP approach for different values of parameter ψ, which govern the curvature of the Kimball (1996) aggregation function (the more ψ is negative the more kinked is the demand function). This parameter controls the degree of non linearity in our model, the accuracy of the EP approach should depend on ψ and we expect to see a deterioration of the accuracy when the magnitude of ψ is increased. This parameter also affect the degree of price rigidity in the model. *Ceteris paribus*, an increase in the magnitude of ψ increases the price rigidity. Consequently, with more negative values of the aggregation technology parameter the model will generate less volatility in inflation, and this will decrease the probability for the ZLB to bind when large deflationary productivity shocks hit the economy. To limit this side effect, we adapt the value of the Calvo (1983) probability ν to keep constant the slope of the Phillips curve. If we linearize the model, this slope (*ie* the reduced form parameter associated to the marginal cost) is given by:

$$s = \frac{\varepsilon - 1}{\varepsilon(1-\psi) - 1}\frac{(1-\nu\beta)(1-\nu)}{\nu}$$

Keeping the slope s constant[9], the value of ν, as a function of ψ, is a root of the following quadratic equation:

$$\mathscr{Q}(\nu) \triangleq \nu^2 - \frac{1+\gamma+\beta}{\beta}\nu + \frac{1}{\beta} = 0$$

with

$$\gamma = s\left(\frac{\varepsilon - 1}{\varepsilon(1-\psi) - 1}\right)^{-1}$$

Because $\mathscr{Q}(0) = {}^{1}\!/\!{}_{\beta} > 0$ and $\mathscr{Q}(1) = -{}^{\gamma}\!/\!{}_{\beta} < 0$, we know that, for all admissible values of the deep parameters, this polynomial equation admits only one root between zero and one (the second root is greater than one and therefore meaningless). In the sequel, we will use this root as the value of the Calvo (1983) probability when we change the value of ψ.

5.2. **Results.** Tables 2, 3 and 4 report the results for Test #1 with different values for the Kimball (1996) curvature parameter. Overall the residuals of the Euler equations are pretty small, except when the radius of the hyper-sphere is equal to .1. But this case is not really relevant with respect the volatility of the endogenous state variables (as illustrated in figure 5 where a 10 percent deviation from the deterministic steady is never observed, even in a sample of 10000 periods). Surprisingly, the accuracy errors reported in these tables are of the same order of magnitude than the stopping criterion we choose in the Newton algorithm used to solve the perfect foresight models[10] (see algorithm 1). This suggest that, in periods where the ZLB is not binding,

[9] For the baseline calibration, the value of the slope is $s = 0.0751$.

[10] We stop the Newton iterations when the residuals of the stacked dynamic equations are smaller than 10^{-5}.

we may improve the accuracy of the EP approach by choosing a smaller stopping criterion when solving the perfect foresight models. Another remarkable result is that the accuracy does not deteriorate when the nonlinearity is more important, *ie* when ψ is increased. As we will clearly understand later, this is very likely caused by the ZLB. Even if we adapt the value of the Calvo (1983) probability to keep constant the slope of the (linearized) Phillips curve, an increase of ψ reduces the probability of a ZLB binding for long periods (the economy hits the ZLB less or more softly).

Table 5 presents the results for test #2, with different values for the Kimball (1996) curvature parameter. We generated a sample of 10000 periods and computed the accuracy errors and various statistics using a subsample of 8000 periods. As we concluded with the previous test, the accuracy errors are not sensibly affected by the curvature of the aggregation function (ψ). Again, we observe that the maximum accuracy error is a decreasing function of the Kimball (1996) curvature parameter. In this model the volatility of the variables is decreased when we increase the nonlinearity. Consequently, by increasing the magnitude of ψ, we reduce the probability of hitting the ZLB. Figure 5.2 plots the accuracy errors on a subsample (spanning the periods 1200-1400) containing an episode where the ZLB binds. The remarkable feature is that the accuracy errors deteriorate considerably (by a factor 100) when the economy is hitting the ZLB.

This unpleasant result, may be explained by looking at the conditional density of consumption, see figure 5.2. When the ZLB is not binding, consumption is distributed around the deterministic steady state, where the nonlinearity induced by the kinked demand function is not important. But when the ZLB is binding, the distribution of consumption is "significantly" shifted on the right (*ie* the deterministic steady state is now in the left tail of the conditional density of consumption). In this case, the endogenous variables visit a region where the demand function is effectively kinked, and the accuracy errors induced by omitting the Jensen inequality are more important. In this model, as shown in figure 4, the ZLB binds only when the productivity shocks are big enough (A_t must be greater than 1.04). This explains why the distribution of consumption conditional on a binding ZLB is shifted on the right. Even in a larger model, where this condition on the efficiency is not necessary for the ZLB to bind, we would again observe a shift on the right of the consumption's distribution, because households do not have incentives to postpone consumption when the interest rate is low.

σ_C	1.50000
η	2.00000
β	0.99700
ν	0.75000
ψ	−0.10000
ϵ	6.00000
γ_π	1.20000
$\pi^\star$	1.00533
$h^\star$	0.33300
ρ_a	0.98000
ρ_b	0.20000
σ_a^2	0.00002
σ_b^2	0.00001

TABLE 1. **Calibration.** The value of ϕ as a function of ϵ and ψ is defined in the text, the value of ξ_h is defined by steady state restrictions. Given the values of the deep parameters and the values of the autoregressive parameters (ρ_a and ρ_b) we choose the size of the innovations to match the volatility of the growth rates of consumption and output and of the inflation and nominal interest rates (using European data).

Radius	Statistics	Equation 5	Equation 6	Equation 7	Equation 8
	max	−4.5625	−3.9928	−4.1874	−5.4621
$r = .0001$	mean	−4.5626	−3.9929	−4.1876	−5.4623
	median	−4.5626	−3.9929	−4.1876	−5.4623
	max	−4.5610	−3.9893	−4.1834	−5.4618
$r = .0010$	mean	−4.5620	−3.9909	−4.1854	−5.4633
	median	−4.5619	−3.9908	−4.1852	−5.4634
	max	−4.5446	−3.9503	−4.1393	−5.4532
$r = .0100$	mean	−4.5552	−3.9696	−4.1617	−5.4740
	median	−4.5542	−3.9680	−4.1595	−5.4749
	max	−2.0021	−1.6198	−1.7798	−2.2374
$r = .1000$	mean	−4.3386	−3.6893	−3.8573	−5.2648
	median	−4.4767	−3.8921	−4.0534	−5.3734

TABLE 2. **Accuracy on a growing sphere.** The Kimball (1996) curvature parameter is set to $\psi = -.1$. The reported statistics (maximum, average and median of the unit free error absolute values expressed in base 10 logarithms) are computed considering 1000 deviates uniformly distributed on an hyper-sphere.

Radius	Statistics	Equation 5	Equation 6	Equation 7	Equation 8
$r = .0001$	max	−4.7086	−4.1386	−3.9080	−6.2203
	mean	−4.7088	−4.1401	−3.9089	−6.2278
	median	−4.7088	−4.1401	−3.9089	−6.2278
$r = .0010$	max	−4.7070	−4.1253	−3.9001	−6.1591
	mean	−4.7088	−4.1401	−3.9089	−6.2292
	median	−4.7088	−4.1400	−3.9088	−6.2273
$r = .0100$	max	−4.6907	−3.9578	−3.8044	−5.7778
	mean	−4.7086	−4.1350	−3.9047	−6.1149
	median	−4.7084	−4.1324	−3.9037	−6.0583
$r = .1000$	max	−2.3512	−1.6323	−1.5055	−2.9112
	mean	−4.3514	−4.0334	−3.6585	−5.4341
	median	−4.6920	−4.1250	−3.9048	−5.6367

TABLE 3. **Accuracy on a growing sphere.** The Kimball (1996) curvature parameter is set to $\psi = -5$. The reported statistics (maximum, average and median of the unit free error absolute values expressed in base 10 logarithms) are computed considering 1000 deviates uniformly distributed on an hyper-sphere.

Radius	Statistics	Equation 5	Equation 6	Equation 7	Equation 8
$r = .0001$	max	−4.6662	−3.6223	−3.4713	−5.5213
	mean	−4.6668	−3.6263	−3.4740	−5.5225
	median	−4.6667	−3.6266	−3.4742	−5.5225
$r = .0010$	max	−4.6649	−3.5699	−3.4362	−5.5113
	mean	−4.6693	−3.6208	−3.4703	−5.5229
	median	−4.6671	−3.6219	−3.4712	−5.5222
$r = .0100$	max	−4.6539	−3.4772	−3.3653	−5.4301
	mean	−4.6673	−3.6237	−3.4707	−5.5273
	median	−4.6660	−3.6239	−3.4724	−5.5211
$r = .1000$	max	−2.6588	−1.6746	−1.5883	−3.3737
	mean	−4.3029	−3.4325	−3.1926	−5.2492
	median	−4.6367	−3.4936	−3.3863	−5.3683

TABLE 4. **Accuracy on a growing sphere.** The Kimball (1996) curvature parameter is set to $\psi = -10$. The reported statistics (maximum, average and median of the unit free error absolute values expressed in base 10 logarithms) are computed considering 1000 deviates uniformly distributed on an hyper-sphere.

ψ	Statistics	Equation 5	Equation 6	Equation 7	Equation 8
$\psi = -.1$	max	-0.9292	-1.1710	-1.2858	-1.1032
	mean	-3.8888	-3.6027	-3.7435	-4.1949
	median	-4.5602	-3.9919	-4.1864	-5.4641
$\psi = -5$	max	-2.1846	-1.5794	-1.4371	-2.7911
	mean	-4.2430	-3.6145	-3.4458	-4.9554
	median	-4.7075	-4.1414	-3.9098	-5.8641
$\psi = -10$	max	-2.5653	-1.6661	-1.5686	-3.3126
	mean	-4.3818	-3.3571	-3.2376	-5.1921
	median	-4.6659	-3.6226	-3.4710	-5.5169

TABLE 5. **Accuracy along a simulated path.** The reported statistics (maximum, average and median of the unit free error absolute values expressed in base 10 logarithms) are computed considering a sample of 8000 simulated data (with the EP approach) containing episodes where the ZLB is binding.

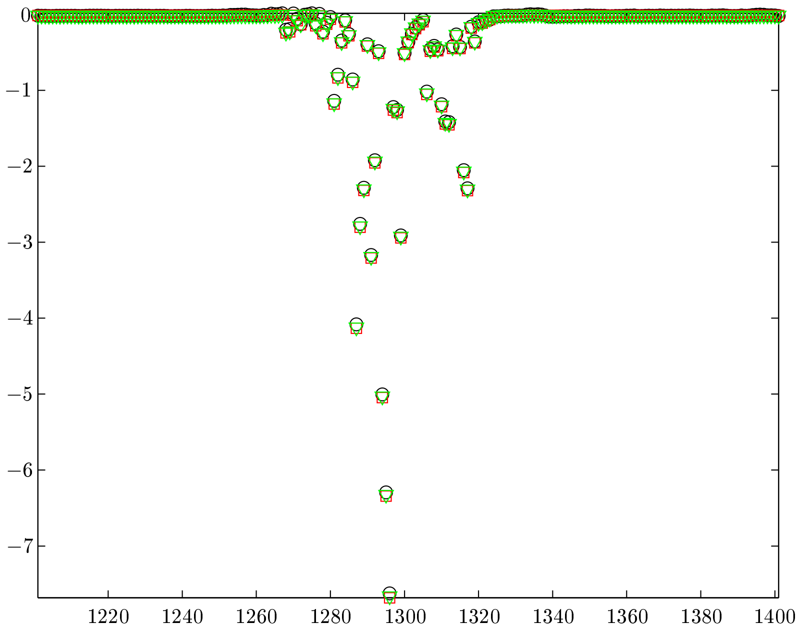

FIGURE 1. **Comparison between EP and perturbations (for output paths, Y_t) when the ZLB is binding.** The deviations are expressed in percentage of the steady state level. Black circles are for the gaps between EP and first order perturbation, red squares are for the gaps between EP and second order perturbation and green triangles are for the gaps between EP and second order perturbation with pruning as advocated by Kim et al. (2008).

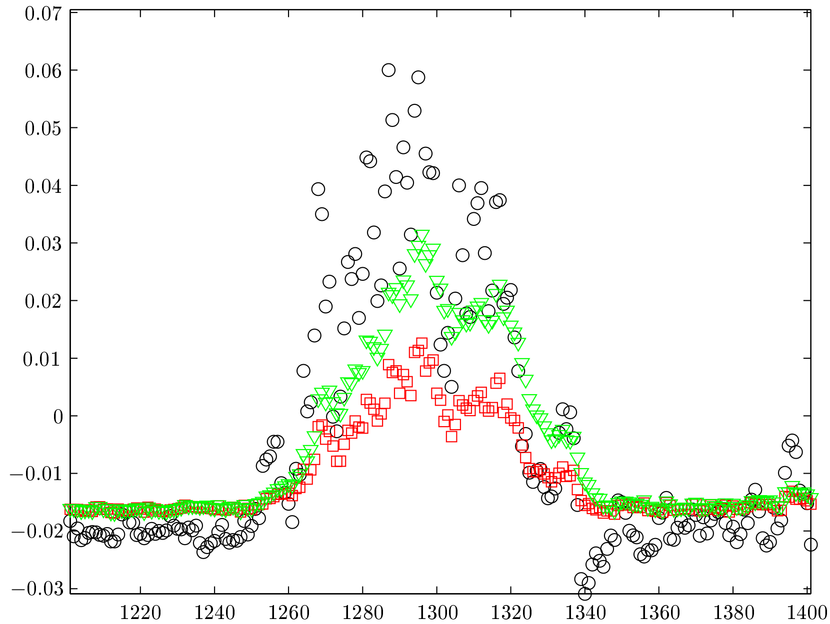

FIGURE 2. **Comparison between EP and perturbations (for output paths, Y_t) when the ZLB is not imposed.** The deviations are expressed in percentage of the steady state level. Black circles are for the gaps between EP and first order perturbation, red squares are for the gaps between EP and second order perturbation and green triangles are for the gaps between EP and second order perturbation with pruning as advocated by Kim et al. (2008).

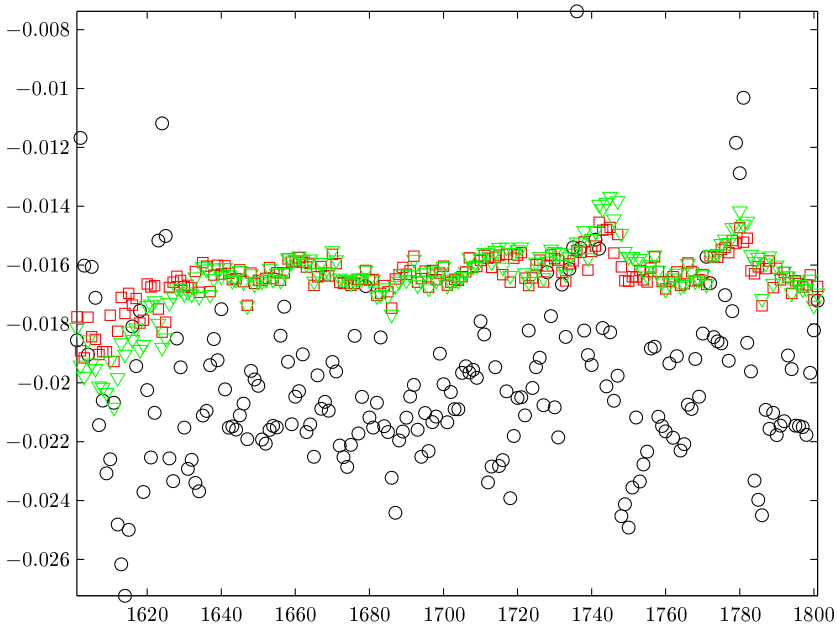

FIGURE 3. **Comparison between EP and perturbations (for output paths, Y_t) when the ZLB is not binding.** The deviations are expressed in percentage of the steady state level. Black circles are for the gaps between EP and first order perturbation, red squares are for the gaps between EP and second order perturbation and green triangles are for the gaps between EP and second order perturbation with pruning as advocated by Kim et al. (2008).

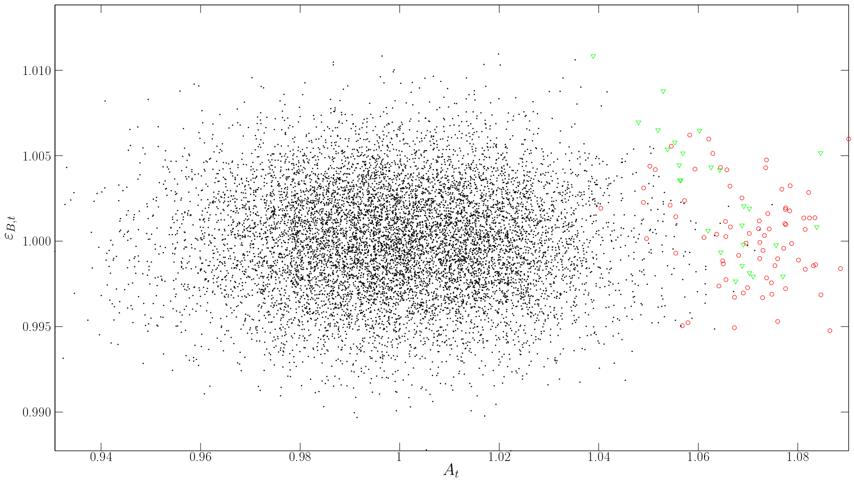

FIGURE 4. **Distribution of the exogenous state variables.** Black dots, red circles and green triangles respectively represent, the levels of efficiency and risk premium when the ZLB constraint is not binding, binding or expected to bind in the future.

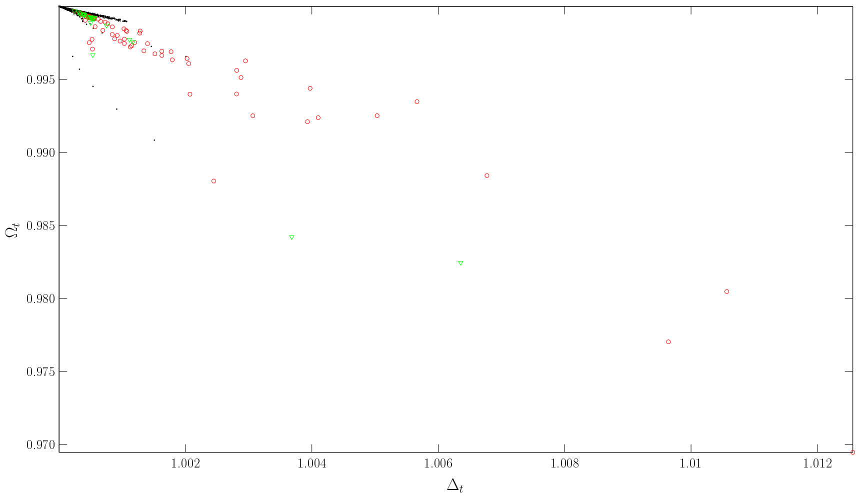

FIGURE 5. **Distribution of the endogenous state variables.** Black dots, red circles and green triangles respectively represent, the levels of the price distorsion Δ_t and the transformed Lagrangian Ω_t when the ZLB constraint is not binding, binding or expected to bind in the future.

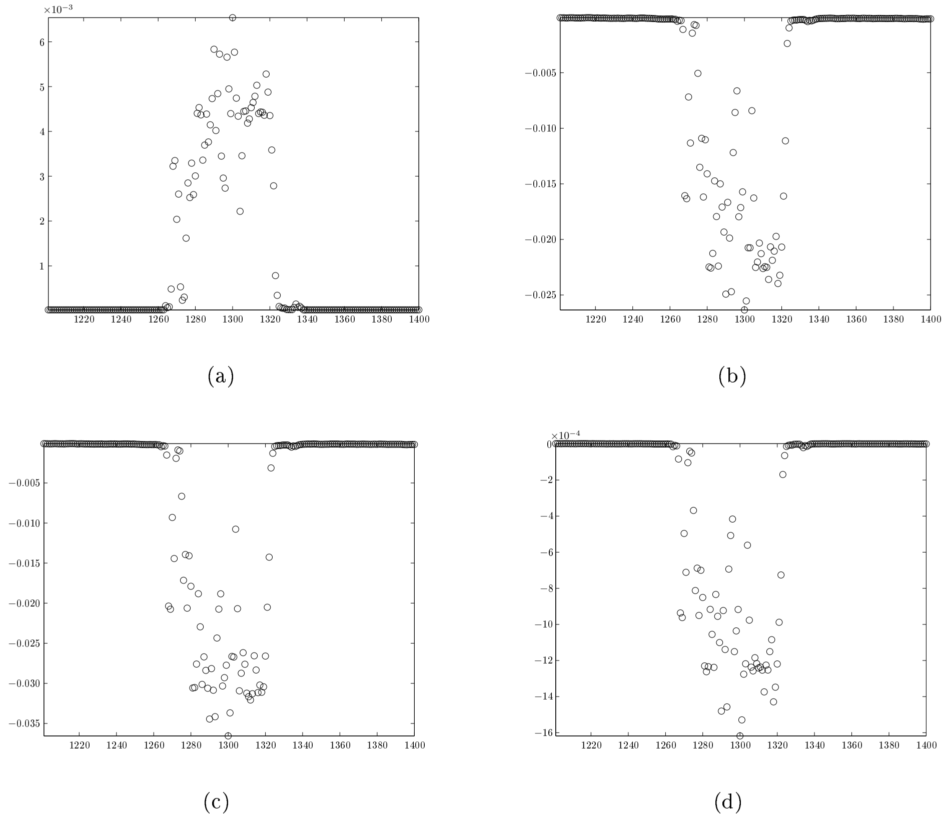

FIGURE 6. **Residuals of the Euler equations along a simulated path.** The Kimball (1996) curvature parameter is set to $\psi = -5$. The reported residuals are unit free, *ie* equations (5) to (8) were respectively divided by λ_t, $Z_{1,t}$, $Z_{2,t}$ and $Z_{3,t}$. We only plot a subsample of 200 periods containing an episode were the ZLB is binding.

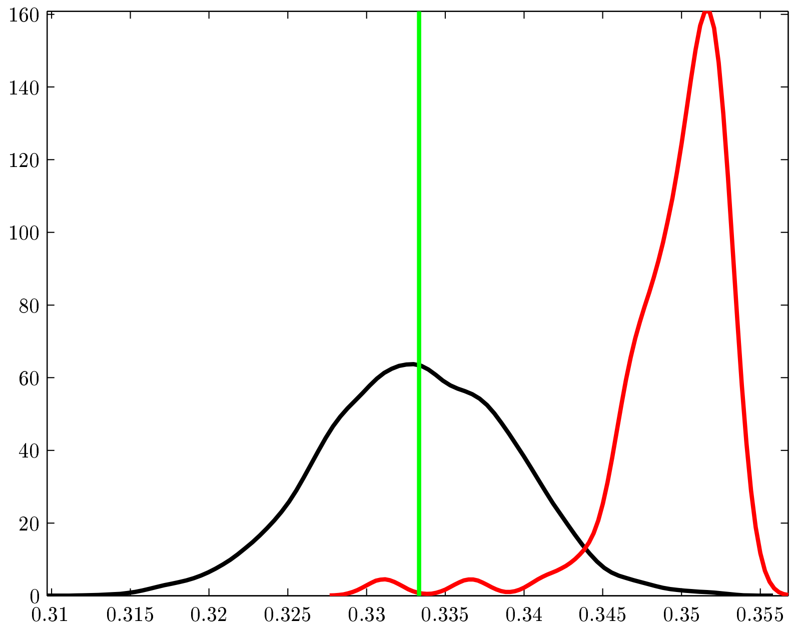

FIGURE 7. **Consumption conditional density.** The dark and red curves respectively represent the non parametric estimates for the density of consumption conditional on "non binding ZLB" and a "binding ZLB". These estimates were obtained using simulated paths of 10000 periods, with the EP algorithm and our baseline calibration. The vertical green line represent the deterministic steady state for consumption.

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Appendix A. Price setting behavior

A.1. **Final good sector.** The first order condition associated to (FGS3) determines the demand for intermediary good z:

$$\frac{Y_t(z)}{Y_t} = \frac{1}{1+\psi}\left[\left(\frac{P_t(z)/P_t}{\Sigma_t}\right)^{-(1+\psi)\varepsilon} + \psi\right] \tag{A.1}$$

where Σ_t is the Lagrange multiplier associated with the technological constraint in (FGS3). Substituting (A.1) in the technological constraint (FGS1)-(FGS2), one gets the following expression for the Lagrange multiplier:

$$\Sigma_t = \left(\int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{1-\varepsilon(1+\psi)} \mathrm{d}z\right)^{\frac{1}{1-\varepsilon(1+\psi)}} \tag{A.2}$$

Equivalently, we have:

$$\Sigma_t^{1-\varepsilon(1+\psi)} = \int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{1-\varepsilon(1+\psi)} \mathrm{d}z$$

The price, $P_t(z)$, appearing under the integral has been set optimally at time $t-j$ with probability $(1-\nu)\nu^j$. Therefore we also have:

$$\Sigma_t^{1-\varepsilon(1+\psi)} = (1-\nu)\sum_{j=0}^{\infty} \nu^j \left(\pi^{\star j}\frac{P_{t-j}^\star}{P_t}\right)^{1-\varepsilon(1+\psi)}$$

and this expression admits the following recursive representation:

$$\Theta_t = (1-\nu)\left(\frac{P_t^\star}{P_t}\right)^{\frac{1}{1-\phi}} + \nu\left(\frac{\pi^\star}{\pi_t}\right)^{\frac{1}{1-\phi}}\Theta_{t-1} \tag{A.3}$$

with $\phi = \frac{\varepsilon(1+\psi)}{\varepsilon(1+\psi)-1}$ and $\Omega_t \triangleq \Sigma_t^{\frac{1}{1-\phi}}$. This proves equation (10). Finally, as the final good sector is perfectly competitive, profit for the representative firm must be zero and we derive the aggregate price index:

$$P_t = \frac{\psi}{1+\psi}\int_0^1 P_t(z)\mathrm{d}z + \frac{1}{1+\psi}\left(\int_0^1 P_t(z)^{1-(1+\psi)\varepsilon}\mathrm{d}z\right)^{\frac{1}{1-(1+\psi)\varepsilon}} \tag{A.4}$$

Comparing the last equation with (A.2) and dividing by P_t we also have:

$$\frac{\psi}{1+\psi}\int_0^1 \frac{P_t(z)}{P_t}\mathrm{d}z + \frac{\Theta_t^{1-\phi}}{1+\psi} = 1$$

where the integral in the first term of the left hand side also admits a recursive representation. Let ϑ_t be the sum of relative prices at time t we have:

$$\vartheta_t = (1-\nu)\left(\frac{P_t^\star}{P_t}\right) + \nu\left(\frac{\pi^\star}{\pi_t}\right)\vartheta_{t-1} \tag{A.5}$$

so that finally we have:

$$\frac{\psi\vartheta_t}{1+\psi} + \frac{\Theta_t^{1-\phi}}{1+\psi} = 1 \tag{A.6}$$

with ϑ_t and Θ_t respectively defined in (A.5) and (A.3). This last equation defines a non linear deterministic relationship between two states variables, ϑ_t and Θ_t. Substituting (A.6) in (A.5) we obtain:

$$1+\psi-\Theta_t^{1-\phi} = \psi(1-\nu)\left(\frac{P_t^\star}{P_t}\right) + \nu\left(\frac{\pi^\star}{\pi_t}\right)\left(1+\psi-\Theta_{t-1}^{1-\phi}\right) \tag{A.7}$$

A.2. **Intermediate goods sector.** Let $\widetilde{\mathscr{V}}_t$ be the value of a firm that receives a positive signal in period t and $\mathscr{V}_t(P_{t-1}(z))$ the value of a firm that receives a negative signal. As a firm that receives a negative signal follows simply the *ad hoc* pricing rule $P_t(z) = \pi^\star P_{t-1}(z)$, its value at time t depends only on $P_{t-1}(z)$. For a firm that receives a positive signal, its value at period t is

$$\widetilde{\mathcal{V}}_t = \max_{\mathbf{P}}\left\{\Pi_t\Big(\mathbf{P}\Big) + \beta\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\Big((1-\nu)\widetilde{\mathcal{V}}_{t+1} + \nu\mathcal{V}_{t+1}(\mathbf{P})\Big)\right]\right\} \tag{A.8}$$

where Λ_t is the Lagrange multiplier of the budget constraint of the representative household and $\Lambda_t = P_t\lambda_t$. Let $P^\star$ be the optimal price choosen by the firm that can re–optimize. The value of a firm that can't re–optimize is

$$\begin{aligned}\mathcal{V}_t\big(P_{t-1}(z)\big) =&\Pi_t\Big(\pi^\star P_{t-1}(z)\Big)\\ &+\beta\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\Big((1-\nu)\widetilde{\mathcal{V}}_{t+1} + \nu\mathcal{V}_{t+1}\big(\pi^\star P_{t-1}(z)\big)\Big)\right]\end{aligned} \tag{A.9}$$

The first order condition and the envelope theorem give:

$$\Pi_t'\big(P^\star\big) + \beta\nu\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\mathcal{V}_{t+1}'\big(P^\star\big)\right] = 0 \tag{A.10a}$$

$$\frac{\mathcal{V}_t'\big(P_{t-1}(z)\big)}{\pi^\star} = \Pi_t'\big(\pi^\star P_{t-1}(z)\big) + \beta\nu\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\mathcal{V}_{t+1}'\big(\pi^\star P_{t-1}(z)\big)\right] \tag{A.10b}$$

with the derivative of profit at $\mathcal{P}$:

$$\begin{aligned}\Pi_t'(\mathcal{P}) \quad &= \quad \frac{1-\varepsilon(1+\psi)}{1+\psi}\left(\frac{\mathcal{P}}{P_t}\right)^{-(1+\psi)\varepsilon}\Sigma_t^{(1+\psi)\varepsilon}Y_t\\ &+\varepsilon\left(\frac{\mathcal{P}}{P_t}\right)^{-(1+\psi)\varepsilon-1}\Sigma_t^{(1+\psi)\varepsilon}mc_tY_t + \frac{\psi}{1+\psi}Y_t\end{aligned} \tag{A.11}$$

Let's write temporarily, in order to simplify notations, $\mathcal{P}$, the price inherited from the past. One can rewrite, one period ahead

$$\mathcal{V}_{t+1}'\big(\mathcal{P}\big) = \pi^\star\Pi_{t+1}'\big(\pi^\star\mathcal{P}\big) + \beta\nu\pi^\star\mathbb{E}_{t+1}\left[\frac{\Lambda_{t+2}}{\Lambda_{t+1}}\mathcal{V}_{t+2}'\big(\pi^\star\mathcal{P}\big)\right]$$

Iterating forward and applying conditional expectation at time t, one gets

$$\mathbb{E}_t\left[\mathcal{V}_{t+1}'\big(\mathcal{P}\big)\right] = \mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\nu)^j\pi^{\star j+1}\frac{\Lambda_{t+1+j}}{\Lambda_{t+1}}\Pi_{t+1+j}'\big(\pi^{\star j+1}\mathcal{P}\big)\right]$$

By substitution ($\mathcal{P} = P^\star$) in the first order condition, one gets the following condition for the price chosen by the firm that gets a positive signal:

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\nu)^j\pi^{\star j}\frac{\Lambda_{t+j}}{\Lambda_t}\Pi_{t+j}'\big(\pi^{\star j}P_t^\star\big)\right] = 0 \tag{A.12}$$

One can get a more explicit expression for the price that satisfies equation (A.12). Substituting in this equation the expression of marginal profit (A.11) and dividing by $P_t^{\star\,-(1+\psi)\varepsilon}$ one gets:

$$\frac{P_t^\star}{P_t} = \phi\frac{\mathscr{Z}_{1,t}}{\mathscr{Z}_{2,t}} + \psi(\phi-1)\left(\frac{P_t^\star}{P_t}\right)^{1+(1+\psi)\varepsilon}\frac{\mathscr{Z}_{3,t}}{\mathscr{Z}_{2,t}} \tag{A.13}$$

with

$$\mathscr{Z}_{1,t} = \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\nu)^j \lambda_{t+j} \left(\frac{\pi^{\star j}}{P_{t+j}/P_t} \right)^{-(1+\psi)\varepsilon} \Sigma_{t+j}^{(1+\psi)\varepsilon} mc_{t+j} Y_{t+j} \tag{A.14a}$$

$$\mathscr{Z}_{2,t} = \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\nu)^j \lambda_{t+j} \left(\frac{\pi^{\star j}}{P_{t+j}/P_t} \right)^{1-(1+\psi)\varepsilon_t} \Sigma_{t+j}^{(1+\psi)\varepsilon} Y_{t+j} \tag{A.14b}$$

$$\mathscr{Z}_{3,t} = \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\nu)^j \lambda_{t+j} \frac{\pi^{\star j}}{P_{t+j}/P_t} Y_{t+j} \tag{A.14c}$$

writing ${}^{P_{t+j}}/{}_{P_t}$, the inflation factor between t and $t+j$, can be written equivalently $\Pi_{i=1}^{j}\pi_{t+i}$, and we can represent variables $\mathscr{Z}_{1,t}$, $\mathscr{Z}_{2,t}$ et $\mathscr{Z}_{3,t}$ in recursive form:

$$\mathscr{Z}_{1,t} = \widehat{\lambda}_t \frac{w_t}{A_t} \Theta_t^{-\phi} \widehat{Y}_t + \beta\nu\mathbb{E}_t \left[\left(\frac{\pi^\star}{\pi_{t+1}} \right)^{\frac{\phi}{1-\phi}} \mathscr{Z}_{1,t+1} \right] \tag{A.15a}$$

$$\mathscr{Z}_{2,t} = \widehat{\lambda}_t \Theta_t^{-\phi} \widehat{Y}_t + \beta\nu\mathbb{E}_t \left[\left(\frac{\pi^\star}{\pi_{t+1}} \right)^{\frac{1}{1-\phi}} \mathscr{Z}_{2,t+1} \right] \tag{A.15b}$$

$$\mathscr{Z}_{3,t} = \widehat{\lambda}_t \widehat{Y}_t + \beta\nu\mathbb{E}_t \left[\left(\frac{\pi^\star}{\pi_{t+1}} \right) \mathscr{Z}_{3,t+1} \right] \tag{A.15c}$$

Appendix B. Steady state of the model

For the exogenous state variables we have:

$$A = A^\star \tag{B.1}$$

$$\varepsilon_B^\star = 1 \tag{B.2}$$

We impose the following steady state levels for inflation and hours:

$$\pi^\star = 1.02 \tag{B.3}$$

$$h^\star = 0.33 \tag{B.4}$$

The long run level of the nominal interest is a function of the discount factor:

$$R^\star = \frac{\pi^\star}{\beta} \tag{B.5}$$

Equations (10) and (11) at the steady states give:

$$(1-\nu)\left[p^{\star\,\frac{1}{1-\phi}} - \Theta^\star \right] = 0$$

$$(1-\nu)\left[\psi p^\star + 1 - \psi - \Theta^{\star\,1-\phi} \right] = 0$$

These two equations are satisfied iff:

$$p^\star = 1 \tag{B.6}$$

$$\Theta^\star = 1 \tag{B.7}$$

Plugging these results in equation (12) we get the steady state level of the price distorsion:

$$\Delta^\star = 1 \tag{B.8}$$

Equations (6) to (8) imply

(a)
$$\mathscr{Z}_1^\star = \frac{mc^\star Y^\star}{1-\nu\beta}$$

where the marginal cost at the steady state is $mc^\star = {}^{w^\star}/_{A^\star}$,

(b)
$$\mathscr{Z}_2^\star = \frac{Y^\star}{1-\nu\beta}$$

and

(c)
$$\mathscr{Z}_3^\star = \frac{Y^\star}{1-\nu\beta}$$

Substituting (a) to (c) in equation (9) at the steady state, we obtain:

$$\frac{\mathscr{Z}_2^\star}{(1+\psi)(1-\phi)} + \frac{\mathscr{Z}_3^\star\psi}{1+\psi} = \frac{\phi\mathscr{Z}_1^\star}{(1+\psi)(1-\phi)}$$
$$\Leftrightarrow \frac{1}{(1+\psi)(1-\phi)} + \frac{\psi}{1+\psi} = \frac{\phi mc^\star}{(1+\psi)(1-\phi)}$$

Substituting the definitions of ϕ and $mc^\star$ in the last equation, we get the steady state level of real wage:

(B.9)
$$w^\star = \frac{\varepsilon-1}{\varepsilon}A^\star$$

Because the effect of the price distortion at the steady state cancels out, equation (13) implies:

(B.10)
$$Y^\star = A^\star h^\star$$

We obtain the steady state level of consumption by equation (14):

(B.11)
$$C^\star = A^\star h^\star$$

and the level of the marginal utility of consumption using equation (3):

(B.12)
$$\lambda^\star = (A^\star h^\star)^{-\sigma_c}$$

The value of the scale parameter ξ_h is determined by the steady state restrictions, *e.g.* equation (4):

(B.13)
$$\xi_h = -\frac{(A^\star h^\star)^{-\sigma_c}\,\frac{\varepsilon-1}{\varepsilon}}{h^{\star\eta}}$$

The steady state of the auxiliary variables $\mathscr{Z}_1$, $\mathscr{Z}_2$, $\mathscr{Z}_3$, $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{E}_3$ and $\mathscr{E}_4$ are obvious.

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